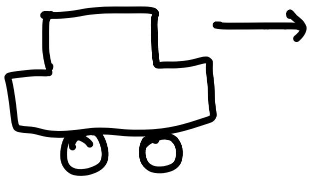
Q1

a)  100 km/h

Let $y(t)$ be the position of a car

Then $y'(t)$ is the velocity and

$y''(t)$ is the accerelation.

we know the car is moving 100 km/h

and the deceleration is $10\ m/s^2$.

$\Rightarrow$ it gives

$$y'(t)\big|_{t=0} = 100\ km/h = 100 \cdot 1000\ m/3600\ s$$

$$\approx 27.78\ m/s$$

$$y''(t) = -10\ m/s^2$$

then we can get the diff eqn that describes the speed of a car.

$y'(t) = y'(t)\big|_{t=0} - 10 \cdot t$

$\Rightarrow \; y'(t) = -10t + 27.78 \;\; (m/s) \cdots ①$

Also, we can get the position of a car by integrating the speed of a car.

$\Rightarrow \; y(t) = y(t)\big|_{t=0} - 5t^2 + 27.78t \;\; (m)$

Assume $y(t)\big|_{t=0}$ is 0.

Then $\; y(t) = -5t^2 + 27.78t \;\; (m) \cdots ②$

b) The car stops when $y'(t) = 0$

$\Rightarrow \; y'(t) = -10t + 27.78 = 0$

$\Rightarrow \; t = 2.778$

Then the position of the car when it stops is $y(2.778)$

$\Rightarrow y(2.778) = -5 \cdot (2.778)^2 + 27.78 \cdot 2.778$

$\approx 38.59 \text{ m}$

Q2

$$\begin{cases} y' = 6e^{2x-y} \\ y(0) = 0 \end{cases}$$

we can rewrite the given equation using the separation of variables.

$$\Rightarrow \quad e^{y} y' = 6e^{2x}$$

$$\Rightarrow \quad e^{y} \frac{dy}{dx} = 6e^{2x}$$

$$\Rightarrow \quad e^{y}\, dy = 6e^{2x}\, dx$$

By integrating the both sides, we get

$$e^{y} = 3e^{2x} + c_1$$

Then $y = \ln(3e^{2x} + c_1)$

By the initial condition, $y(0) = 0$,
we can easily get c_1.

$$\Rightarrow \quad 0 = \ln(3 + c_1)$$

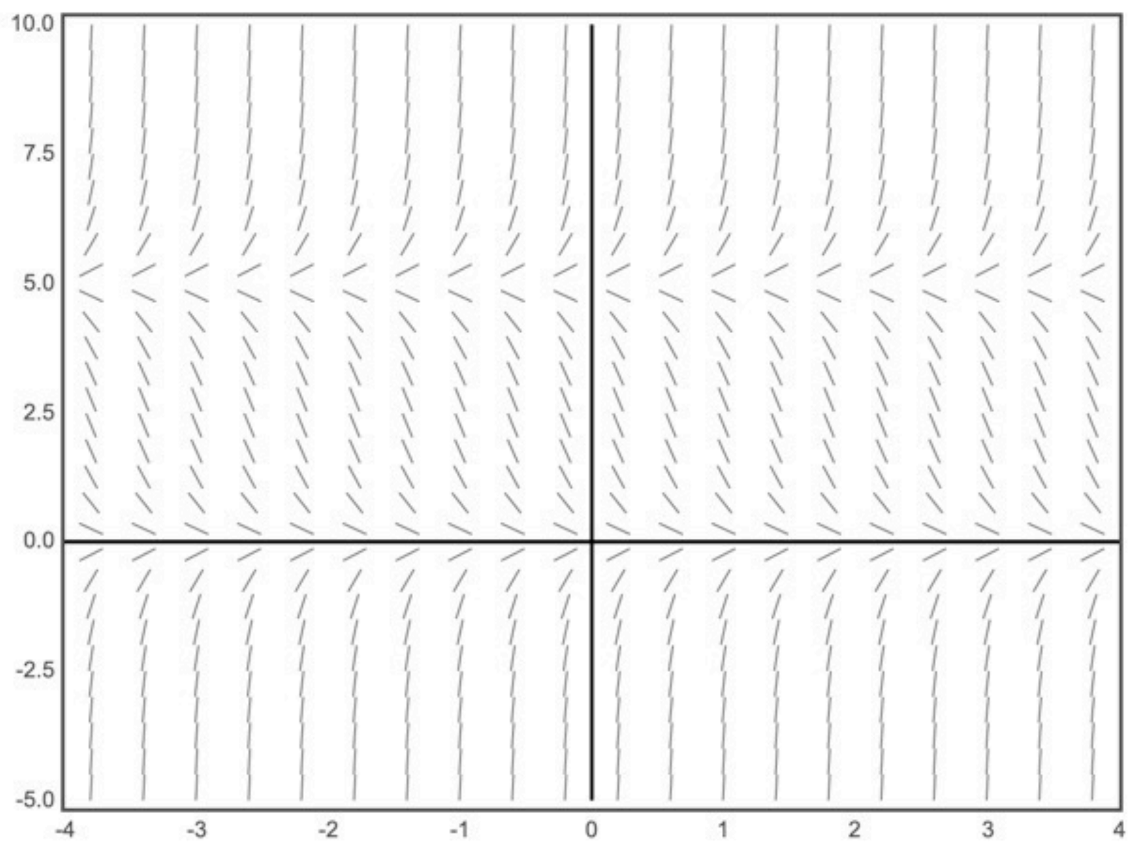
$$\Rightarrow \quad c_1 = -2$$

Thus, the solution is

$$y(x) = \ln(3e^{2x} - 2)$$

Q3

a) $y' = y(y-5)$



Q4. $\begin{cases} y' = 2xy + 3x^2 e^{x^2} \\ y(0) = 5 \end{cases}$

$$\Rightarrow \quad e^{-x^2} y' - 2x e^{-x^2} y = 3x^2$$

The LHS can be written as

$$(e^{-x^2} y)' = e^{-x^2} y' - 2x e^{-x^2} y$$

$$\Rightarrow \quad (e^{-x^2} y)' = 3x^2$$

By integrating both sides, we get

$$e^{-x^2} y = x^3 + C$$

$$\Rightarrow \quad y = x^3 e^{x^2} + C e^{x^2}$$

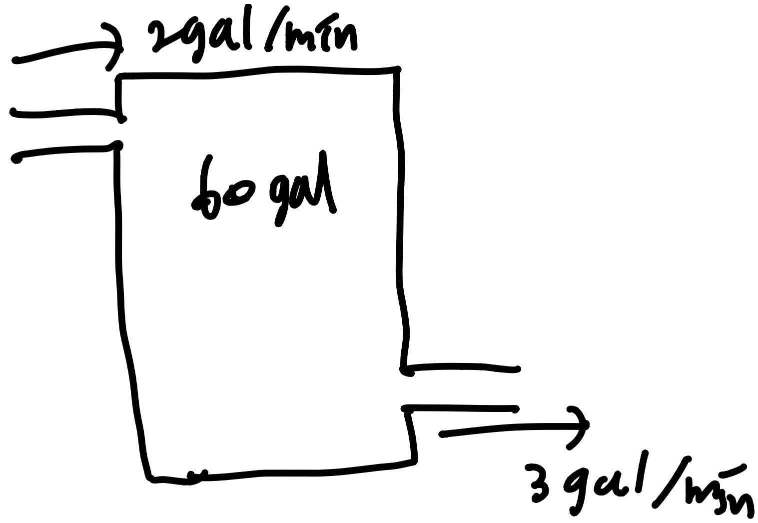
The initial condition gives

$$y(0) = 5 \Rightarrow C = 5$$

Thus, the solution to the given differential equation is

$$y(x) = x^3 e^{x^2} + 5 e^{x^2}.$$

Q5



Let $S(t)$ be the amount of salt in the tank.

$$\frac{dS(t)}{dt} = S_{in} - S_{out}$$

$$= 2 - \frac{S(t)}{60-t} \cdot 3$$

$$\Rightarrow \begin{cases} S' = 2 - \dfrac{3S}{60-t} \\ \underline{S(0) = 0} \end{cases}$$

Initially, the tank only contains pure water

To solve the given equation, we use the integrating factor.

$$S' + \frac{3S}{60-t} = 2$$

$$\Rightarrow \mu(t) = c\, e^{\int \frac{3}{60-t} dt}$$

$$= c\, e^{-3 \ln(60-t)}$$

$$= c\, (60-t)^{-3}$$

$$\Rightarrow S' c(60-t)^{-3} + \frac{3S}{(60-t)} \cdot c(60-t)^{-3}$$

$$= 2c\,(60-t)^{-3}$$

$$\Rightarrow \left(S\,(60-t)^{-3}\right)' = 2\,(60-t)^{-3}$$

$$\Rightarrow S(60-t)^{-3} = (60-t)^{-2} + c$$

$$\Rightarrow S = (60-t) + c\,(60-t)^{3}$$

$S(0)=0 \Rightarrow c = -\frac{1}{60^2}$

Thus, $S(t) = (60-t) - \frac{1}{3600}(60-t)^3$

b) the maximum amount of salt is

when $\frac{ds}{dt} = 0$ & $\frac{d^2s}{dt^2} < 0$

$$\frac{ds}{dt} = -1 + \frac{1}{1200}(60-t)^2 = 0$$

$$\Rightarrow (60-t)^2 = 1200$$

$$\Rightarrow 60-t = \pm\sqrt{1200} = \pm 20\sqrt{3}$$

$$\Rightarrow t = 60 \mp 20\sqrt{3}$$

$$\frac{d^2s}{dt^2} = -\frac{1}{600}(60-t)$$

If $t = 60-20\sqrt{3}$, $\frac{d^2s}{dt^2} < 0$

If $t = 60+20\sqrt{3}$, $\frac{d^2s}{dt^2} > 0$

Thus, when $t = 60 - 20\sqrt{3}$,

the tank has the maximum amount of salt.

Or this can be easily verified by the condition $t \leq 60$.

b) $S_{max} = S(60 - 20\sqrt{3})$

$$= 20\sqrt{3} - \frac{1}{3600}(20\sqrt{3})^3$$

$$= 20\sqrt{3} - \frac{1}{3}20\sqrt{3}$$

$$= \frac{40\sqrt{3}}{3} \text{ lbs}$$