Problem 1: Prove that if \( \lim_{n \to \infty} x_n = x \), and \( x > 0 \) then there exists \( M \in \mathbb{N} \) such that \( x_n > 0 \) for all \( n > M \).

Solution:

Choose \( \varepsilon \) such that \( 0 < \varepsilon < x \).

Then, by definition of limit, \( \exists \ M \in \mathbb{N} \) such that

\[
| x_n - x | < \varepsilon \quad \text{if} \quad n > M.
\]

which means that

\[
x - \varepsilon < x_n < \varepsilon + x \quad \text{for all} \quad n > M.
\]

As \( x - \varepsilon > 0 \) for our choice of \( \varepsilon \),

we obtain that \( x_n > 0 \) for all \( n > M \).
Problem 2: If $f$ is a continuous function on an interval $[a,b]$ and $|f(x)| = 1$ for all $x \in [a,b]$ show that $f$ is a constant function.

Solution:

$|f(x)| = 1 \implies f(x)$ can only take values $\pm 1$. Assume $f$ is not a constant function, then we can find two points $x_0, x_1 \in [a,b]$ such that $f(x_0) = 1$, $f(x_1) = -1$.

As $f$ is continuous, it then takes all intermediate values from $-1$ to $+1$ on the interval $[x_0, x_1]$, which is contradiction.
Problem 3:

Let \( (a_n) \) and \( (b_n) \) be sequences of real numbers. Suppose \( b_n \) is monotonically decreasing \( \lim_{n\to\infty} b_n = 0 \) and \( |a_n| = b_n - b_{n+1} \) for all \( n \in \mathbb{N} \). Prove that \( \sum_{n=1}^{\infty} a_n \) converges absolutely.

Solution:

\[ \sum_{n=1}^{\infty} a_n \text{ converges absolutely if } \sum_{n=1}^{\infty} |a_n| \text{ converges.} \]

Partial sum \( S_n = |a_1| + \ldots + |a_n| \), and \( |a_{n+1}| = b_n - b_{n+1} \Rightarrow \)

\[ S_n = (b_1 - b_2) + (b_2 - b_3) + \ldots + (b_n - b_{n+1}) = b_1 - b_{n+1} \]

Clearly \( \lim_{n \to \infty} S_n = b_1 - \lim_{n \to \infty} b_{n+1} = b \).

Problem 4:

Suppose \( (x_n) \) is a sequence that is bounded and such that \( x_1, x_2, x_3, \ldots \) are distinct. Assume that the set \( \{x_n\} \) has just one limit point \( x \). Prove that the sequence is convergent and its limit is \( x \).
**Solution:**

Assume $x_n$ does not converge to $x$.

Fix $\varepsilon > 0$. Then for each $k \in \mathbb{N}$ we can find $x_{nk} \in \{x_n\}$ such that

$$
(*) \quad |x_{nk} - x| > \varepsilon, \quad \text{for } nk > k.
$$

Indeed, if such $x_{nk}$ doesn't exist then

$$
|x_n - x| < \varepsilon \quad \text{for all } n > k
$$

which means that $(x_n)$ converges to $x$. Thus we have a subsequence $(x_{nk})$.

As $(x_n)$ is bounded, $(x_{nk})$ is also bounded. By W.T. $(x_{nk})$ has convergent subsequence and we denote its limit $y$. Note that $x \neq y$

We conclude that set $\{x_n\}$ has at least 2 limit points $x$ and $y$. \(\Rightarrow\) Contradiction.
Problem 5:

Assume that $f$ is differentiable for all $x \in \mathbb{R}$. If $f(0)=1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \leq x+1$ for all $x > 0$.

**Solution:** By contradiction, assume there is $x_0 > 0$ such that $|f(x_0)| > x_0 + 1$. We have two cases:

1) $f(x_0) > x_0 + 1$ or 2) $f(x_0) < -x_0 - 1$

In first case, by intermediate value theorem for derivative, there is a point $c > 0$ such that

$$f'(c) = \frac{f(x_0) - f(0)}{x_0 - 0} > \frac{x_0 + 1 - 1}{x_0} = 1$$

$\Rightarrow |f'(c)| > 1 \Rightarrow \text{Contradiction}$

In second case, similarly

$$f'(c) = \frac{f(x_0) - f(0)}{x_0 - 0} < \frac{-x_0 - 1 - 1}{x_0}$$

As the last number is negative, we obtain

$$|f'(c)| > \frac{x_0 + 2}{x_0} > 1$$

Contradiction
Problem 6

Let \( f(x) = \sum_{n=1}^{\infty} \frac{1}{x^2+n^2} \)

(a) Show that \( f(x) \) is continuous.

(b) Show that \( f(x) \) is differentiable and compute its derivative (Hint: \( 2|\text{Im} + 2|x|n \leq x^2 + n^2 \))

Solution:

(a) Note \( \frac{1}{x^2+n^2} \) is continuous for any \( n=1, 2, ... \)

Note that \( \left| \frac{1}{x^2+n^2} \right| \leq \frac{1}{n^2} \)

and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges. Thus by Weierstrass M-test the sum converges uniformly. By Continuous Limit Theorem \( f(x) \) is also continuous.

(b) First note that \( (1x - n)^2 \geq 0 \)

so \( x^2 + n^2 - 2|x|n \geq 0 \) \( \Rightarrow x^2 + n^2 \geq 2|x|n \).

Next:

\[
\left| \left( \frac{1}{x^2+n^2} \right)' \right| = \left| -\frac{2x}{(x^2+n^2)^2} \right| = \frac{2|x|n}{n(x^2+n^2)^2} \leq \frac{1}{n(x^2+n^2)} \leq \frac{1}{n^3}
\]

Thus, by Weierstrass M-test, the sum
\[
\sum_{n=1}^{\infty} \left( \frac{1}{x^2+n^2} \right) \text{ converges uniformly.}
\]

Applying the theorem about uniform convergence and differentiation, we conclude that

\[ f(x) \text{ is differentiable and } \]

\[ f'(x) = -\sum_{n=1}^{\infty} \frac{2x}{(x^2+n^2)^2} \]
Problem 7. Let us define $x \in \mathbb{R}$ to be a boundary point of a set $A$ if every neighborhood of $x$ contains a point in $A$ and $A^c$. Let $\partial A$ denote the set of all boundary points of $A$. Show that $A$ is open if and only if $\partial A \cap A = \emptyset$.

Solution: ($\Rightarrow$)
Assume $\partial A \cap A = \emptyset$ then every $x \in A$ has a neighborhood which is contained in $A$. Indeed, if for some $x \in A$ it is not possible to find a neighborhood which is entirely in $A$, then every neighborhood of $A$ contains points in $A^c$ which contradicts $\partial A \cap A = \emptyset$.

$\Rightarrow$ $A$ is open.

($\Leftarrow$)
Assume $A$ is open, then, by definition, each $x \in A$ has a neighborhood $V(x)$ such that $V(x) \subseteq A$ and thus $V(x) \cap A^c = \emptyset$. This means that $x \notin \partial A$. As this is true for all $x \in A$ we conclude that $\partial A \cap A = \emptyset$. 
Problem 8: Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} 5x, & x \in \mathbb{Q} \\ x^2 + 6, & x \notin \mathbb{Q} \end{cases}$$
Prove that $f$ is discontinuous at $x=1$ and continuous at $x=2$.

Solution: $x=1$:

By definition $f(1) = 5$. If $f$ is continuous at $x=1$ then $\lim f(x_n) = 5$ for any sequence $(x_n)$ such that $\lim x_n = 1$. Consider irrational sequence, for instance $x_n = 1 + \frac{\pi}{n}$ we have $\lim x_n = 1$

but $\lim f(x_n) = 7 \Rightarrow f$ is not cont. at $x=1$.

$x=2$:

As $5x$ and $x^2 + 6$ (as functions on $\mathbb{R}$) are both continuous, for $\varepsilon > 0$ we can find $\delta_1^\varepsilon$ and $\delta_2^\varepsilon$ such that
$$|10 - 5x| < \varepsilon \quad \text{if} \quad |x-2| < \delta_1^\varepsilon$$
$$|10 - (x^2 + 6)| < \varepsilon \quad \text{if} \quad |x-2| < \delta_2^\varepsilon$$

Choose $\delta = \min(\delta_1^\varepsilon, \delta_2^\varepsilon)$ then clearly
$$|10 - f(x)| < \varepsilon \quad \text{if} \quad |x-2| < \delta$$
**Problem 9:** Show that \((a, +\infty)\) has the same cardinality as \(\mathbb{R}\).

**Solution:**
The function \(f : (a, +\infty) \to \mathbb{R}\)
\[
x \mapsto \ln |x - a|
\]
does the trick.

**Problem 10**
Given sets \(A\) and \(B\), define
\[
A + B = \{a + b : a \in A, b \in B\}
\]
Prove that \(\sup (A + B) = \sup (A) + \sup (B)\).

**Solution:** By definition of supremum
\[
a \leq \sup (A) \quad \forall a \in A
\]
\[
b \leq \sup (B) \quad \forall b \in B
\]

Thus
\[
a + b \leq \sup (A) + \sup (B) \quad \forall a + b \in A + B
\]

Thus, \(\sup (A) + \sup (B)\) is an upper bound of \(A + B\) and
\[
\sup (A + B) \leq \sup (A) + \sup (B) \quad (\star)
\]

Now, by definition of \(\sup\):
\[
a + b \leq \sup (A + B) \quad \forall a \in A, b \in B
\]
thus
\[
a \leq \sup (A + B) - b \quad \forall a \in A, b \in B
\]
This means that \(\sup (A + B) - b\) is an upper bound.
of $A$ for any $b \in B$, thus
\[ \sup(A) \leq \sup(A+B) - b \]
In turn, this implies that
\[ b \leq \sup(A+B) - \sup(A) \quad \forall b \in B \]
thus \( \sup(A+B) - \sup(A) \) is an upper bound for \( B \) and so
\[ \sup(B) \leq \sup(A+B) - \sup(A) \]
Which means that \( \sup(A) + \sup(B) \leq \sup(A+B) \)
Together with \((\star)\) we conclude
\[ \sup(A+B) = \sup(A) + \sup(B) \]