Final exam solutions

Problem 1: Prove that if $\lim_{n\to\infty} x_n = x$, and $x > 0$ then there exists $M \in \mathbb{N}$ such that $x_n > 0$ for all $n > M$.

Solution:

Chose ϵ such that $0 < \epsilon < x$.

Then, by definition of limit $\exists\, M \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon \qquad \text{if} \quad n > M.$$

which means that

$$x - \epsilon < x_n < \epsilon + x \qquad \text{for all} \quad n > M.$$

As $x - \epsilon > 0$ for our choice of ϵ we obtain that $x_n > 0$ for all $n > M$.

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Problem 2: If f is a continuous function on an interval $[a,b]$ and $|f(x)| = 1$ for all $x \in [a,b]$ show that f is a constant function.

Solution:

$|f(x)| = 1 \implies f(x)$ can only take values ± 1.
Assume f is not a constant function, then we can find two points $x_0, x_1 \in [a,b]$ such that $f(x_0) = 1$, $f(x_1) = -1$.

As f is continuous, it then takes all intermediate values from -1 to $+1$ on the interval $[x_0\ x_1]$, which is contradiction

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Problem 3:

Let (a_n) and (b_n) be sequences of real numbers. Suppose b_n is monotonically decreasing $\lim b_n = 0$, and $|a_n| = b_n - b_{n+1}$ for all $n \in \mathbb{N}$. Proove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Solution:

$\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. Partial sum $S_n = |a_1| + \dots + |a_n|$, and $|a_n| = b_n - b_{n+1} \Rightarrow$

$$S_n = (b_1 - b_2) + (b_2 - b_3) + \dots + (b_n - b_{n+1}) = b_1 - b_{n+1}$$

Clearly $\lim_{n\to\infty} S_n = b_1 - \lim_{n\to\infty} b_{n+1} = b_1$

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Problem 4:

Suppose (x_n) is a sequence that is bounded and such that $x_1, x_2, x_3 \dots$ are distinct. Assume that the set $\{x_n\}$ has just one limit point x. Proove that the sequence is convergent and its limit is x.

Solution:

Assume x_n does not converge to x.
Fix $\epsilon > 0$. Then for each $k \in \mathbb{N}$ we can find $x_{n_k} \in \{x_n\}$ such that

$$(*) \qquad |x_{n_k} - x| > \epsilon, \quad \text{for } n_k > k.$$

Indeed, if such x_{n_k} doesn't exist then

$$|x_n - x| < \epsilon \quad \text{for all } n > k.$$

which means that (x_n) converges to x.
Thus we have a subsequence (x_{n_k}).
As (x_n) is bounded (x_{n_k}) is also bounded. By W.T. (x_{n_k}) has convergent subsequence and we denote its limit y. Note that $x \neq y$ by $(*)$.
We conclude that set $\{x_n\}$ has at least 2 limit points x and y.
$\Rightarrow$ Contradiction.

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Problem 5:

Assume that f is differentiable for all $x \in \mathbb{R}$.. If $f(0)=1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Prove that $|f(x)| \leq x+1$ for all $x > 0$.

Solution: By contradiction, assume there is $x_0 > 0$ such that $|f(x_0)| > x_0+1$. We have two cases:

1) $f(x_0) > x_0+1$ or 2) $f(x_0) < -x_0-1$

In first case, by intermediate value theorem for derivative, there is a point $c > 0$ such that

$$f'(c) = \frac{f(x_0) - f(0)}{x_0 - 0} > \frac{x_0+1-1}{x_0} = 1$$

$\Rightarrow |f'(c)| > 1 \Rightarrow$ Contradiction

In second case, similarly

$$f'(c) = \frac{f(x_0) - f(0)}{x_0 - 0} < \frac{-x_0-1-1}{x_0}$$

As the last number is negative, we obtain

$$|f'(c)| > \frac{x_0+2}{x_0} > 1$$

Contradiction $\blacksquare$

Problem 6

Let
$$f(x) = \sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$$

(a) Show that $f(x)$ is continuous.

(b) Show that $f(x)$ is differentiable and compute its derivative (Hint $2|x|n \leq x^2+n^2$)

Solution:

(a) note $\frac{1}{x^2+n^2}$ is continuous for any $n=1,2,\ldots$

note that $\left|\frac{1}{x^2+n^2}\right| \leqslant \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Thus by Weierstrass M-test the sum converges uniformly. By Continuous limit theorem $f(x)$ is also continuous.

(b) First note that $(|x|-n)^2 \geqslant 0$

so $x^2+n^2-2|x|n \geqslant 0 \implies x^2+n^2 \geqslant 2|x|n.$

Next:
$$\left|\left(\frac{1}{x^2+n^2}\right)'\right| = \left|-\frac{2x}{(x^2+n^2)^2}\right| = \frac{2|x|n}{n(x^2+n^2)^2} \leqslant \frac{1}{n(x^2+n^2)} \leqslant \frac{1}{n^3}$$

thus, by Weierstrass M-test, the sum

$$\sum_{n=1}^{\infty} \left(\frac{1}{x^2+n^2} \right)' \text{ converges uniformly.}$$

Applying the theorem about uniform convergence and differentiation we conclude that $f(x)$ is differentiable and

$$f'(x) = -\sum_{n=1}^{\infty} \frac{2x}{(x^2+n^2)^2}$$

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<u>Problem 7.</u> Let us define $x \in \mathbb{R}$ to be a boundary point of a set A if every neighborhood of x contains a point in A and A^c. Let ∂A denote the set of all boundary points of A.
Show that A is open if and only if $\partial A \cap A = \emptyset$.

<u>Solution:</u> $(\Rightarrow)$
Assume $\partial A \cap A = \emptyset$ then every $x \in A$ has a neighborhood which is contained in A. Indeed, if for some $x \in A$ it is not possible to find a neighborhood which is entirely in A, then every neighborhood of A contains points in A^c thus $x \in \partial A$ which contradicts $A \cap \partial A = \emptyset$.

$\Rightarrow$ A is open.

$(\Leftarrow)$

Assume A is open, then, by definition, each $x \in A$ has a neighborhood $V(x)$ such that $V(x) \subseteq A$ and thus $V(x) \cap A^c = \emptyset$.
This means that $x \notin \partial A$.
As this is true for all $x \in A$ we conclude that $A \cap \partial A = \emptyset$

Problem 8: Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 5x, & x \in \mathbb{Q} \\ x^2+6, & x \notin \mathbb{Q} \end{cases}$$

Prove that f is discontinuous at $x=1$ and continuous at $x=2$.

Solution: $x=1$:

By definition $f(1)=5$. If f is continuous at $x=1$ then $\lim f(x_n)=5$ for any sequence (x_n) such that $\lim x_n=1$. Consider irrational sequence, for instance $x_n = 1+\frac{\pi}{n}$ we have $\lim x_n = 1$ but $\lim f(x_n)=7 \Rightarrow f$ is not cont. at $x=1$.

$x=2$:

As $5x$ and x^2+6 (as functions on $\mathbb{R}$) are both continuous, $\forall\, \epsilon>0$ we can find δ_ϵ^1 and δ_ϵ^2 such that

$$|10-5x| < \epsilon \quad \text{if} \quad |x-2| < \delta_\epsilon^1$$

$$|10-(x^2+6)| < \epsilon \quad \text{if} \quad |x-2| < \delta_\epsilon^2$$

choose $\delta = \min(\delta_\epsilon^1, \delta_\epsilon^2)$ then, clearly

$$|10-f(x)| < \epsilon \quad \text{if} \quad |x-2| < \delta$$

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Problem 9: Show that $(a, +\infty)$ has the same cardinality as $\mathbb{R}$.

Solution:
The function
$$f : (a, +\infty) \longrightarrow \mathbb{R}$$
$$x \longrightarrow \ln(x-a)$$
does the trick $\blacksquare$

Problem 10

Given sets A and B, define
$$A+B = \{a+b : a \in A,\ b \in B\}$$
Prove that $\sup(A+B) = \sup(A) + \sup(B)$.

Solution: By definition of supremum
$$a \leq \sup(A) \qquad \forall\, a \in A$$
$$b \leq \sup(B) \qquad \forall\, b \in B$$
Thus
$$a+b \leq \sup(A) + \sup(B) \qquad \forall\ a+b \in A+B$$
Thus, $\sup(A) + \sup(B)$ is an upper bound of $A+B$
and $\quad \sup(A+B) \leq \sup(A) + \sup(B)$. $(*)$

Now, by definition of sup:
$$a+b \leq \sup(A+B) \qquad \forall\, a \in A,\ b \in B$$
thus
$$a \leq \sup(A+B) - b \qquad \forall\, a \in A,\ b \in B$$
This means that $\sup(A+B) - b$ is an upper bound

of A for any $b \in B$, thus

$$\sup(A) \leq \sup(A+B) - b$$

In turn, this implies that

$$b \leq \sup(A+B) - \sup(A) \qquad \forall\, b \in B$$

thus $\sup(A+B) - \sup(A)$ is an upper bound of B and so

$$\sup(B) \leq \sup(A+B) - \sup(A)$$

Which means that $\sup(A) + \sup(B) \leq \sup(A+B)$
Together with (*) we conclude

$$\sup(A+B) = \sup(A) + \sup(B)$$

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