Lecture 2: Completeness of $\mathbb{R}$

Rational numbers $\mathbb{Q} = \left\{ a/b \,,\ a, b \in \mathbb{N},\ b \neq 0 \right\}$

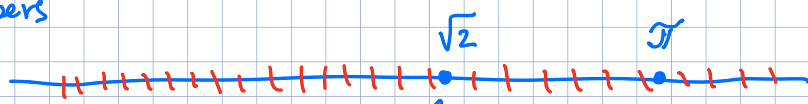
we have $+$, $\cdot$, $<$

real numbers

$\mathbb{R}$

$\cup$

$\mathbb{Q}$



cannot be represented as a/b

$\Rightarrow \sqrt{2} \notin \mathbb{Q}$, $\pi \notin \mathbb{Q}$.

Today: What does it mean ~~th~~ "$\mathbb{R}$ is complete" or "has no holes".

Axiom of completeness:

Every non-empty subset of Real numbers $A \subseteq \mathbb{R}$ that is bounded above has a least upper bound.

?

?

Doesn't hold for $\mathbb{Q}$.

Def: A set $A \subseteq \mathbb{R}$ is bounded from above if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$.

In this case, b is called an upper bound of A.

Example: $I = (0,1) \subseteq \mathbb{R}$

$b=1$ – an upper bound b.c. $a \in I \Rightarrow a \leq 1$

$b=7$ – an upper bound b.c. $a \in I \Rightarrow a \leq 7$.

$\Rightarrow$ upper bound is not unique.

$I = (0, +\infty)$ – no upper bound

Def: A set $A \subseteq \mathbb{R}$ is bounded from below if there exists a number $\ell \in \mathbb{R}$ such that $\ell \leq a$ for all $a \in A$.

"ℓ = lower bound of A"

Ex: $I = (0,1)$

0 – is OK.

-1 – is OK

– – – –

$I = (-\infty, 0)$ – there are no lower bounds.

Def: A Real number $s \in \mathbb{R}$ is the least upper bound of a set $A \subseteq \mathbb{R}$ if:

(1) s is an upper bound.
(2) if b is any ~~not~~ other upper bound then $s \leq b$.

Notation: $s = \sup(A)$

Example:

$I = (0,1)$ ← $\sup(I) = 1$

$I = [0,1]$ ← $\sup(I) = 1$

sup(I) may or may not be an elem. of I.

$I = \{1 - \frac{1}{n^2},\ n = 1,2,3 \dots\}$

$\{0,\ 1-\frac{1}{4},\ 1-\frac{1}{9}, \dots\}$

$\{0,\ \frac{3}{4},\ \frac{8}{9}, \dots\}$ ← $\sup(I) = 1.$

1 is an upper bound

let $b = \sup(I)$, assume $b \neq 1$

$\Rightarrow$ then $b < 1$ then we can

find N large enoug so that

$b < 1 - \frac{1}{N^2}$

$\Rightarrow$ Contradiction because $1 - \frac{1}{N^2} \in I$

but $1 - \frac{1}{N^2} > b = \sup(I)$. $\blacksquare$.

Def: "Greatest lower bound"

— . . . .

$\Rightarrow$ **Notation:** $\inf(A)$

Example: $I = (0,1) \Rightarrow \inf(I) = 0$.

$I = (0,1)$

$I = [0,1)$

$\Rightarrow \inf(I) = 0$

Def: $a_0 \in \mathbb{R}$ is a maximum of a set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$.

Def: $a \in \mathbb{R}$ is a minimum ....

Example:

$I = (0,2) \rightarrow \inf(I)=0, \sup(I)=2,$ no max

$I = [0,2)$ $\inf(I)=0, \sup(I)=2,$ no max

$I = [0,2]$ $\inf(I)=0, \sup(I)=2$

$\max(I)=2.$

$$I = \{x \in \mathbb{R} : x^2 < 2\} = (-\sqrt{2}, +\sqrt{2})$$

$\sup(I) = \sqrt{2}$

$\inf(I) = -\sqrt{2}$

$\min(I) =$ DNE

$\max(I) =$ DNE

Axiom of completeness:

Every non-empty subset of ~~Real~~ rational numbers $A \subseteq \cancel{\mathbb{R}}\ \mathbb{Q}$. that is bounded above has a least upper bound. $S \in \mathbb{Q}$.

$$A = \{x \in \mathbb{Q} : x^2 < 2\} \leftarrow \sqrt{2} \notin \mathbb{Q}.$$

$\Rightarrow$ Dosn't hold for $\mathbb{Q}$

but it holds in $\mathbb{R}$

Ex: Prove that if a is an element of A and is also an upper bound of A then it must be that $a = \sup(A)$.

Proof: A – is bounded from above

$\Rightarrow$ By axiom of completeness there exists $b \in \mathbb{R}$ $b = \sup(A)$.

b is the least upper bound so, by def. of l.u.b we ~~know~~ know that

$$x \leq b \quad \forall x \in A.$$

$\Rightarrow$ in particular $a \leq b$.

As a – upp. bound
b – $\sup(A)$

from definition of l.u.b
if a is any other upper bound

$$b \leq a$$

$\Rightarrow \quad a = b \quad \blacksquare$

Ex: Prove that if a set $A \subseteq \mathbb{R}$ has a least upper Bound that it is unique.

Proof: Assume ~~th~~ least upper bound of A is not unique.

$S_1, S_2 \in \mathbb{R}$ so that $S_1 = \sup(A)$, $S_2 = \sup(A)$.

As S_1 is l.u.b and S_2 is other u.b

$$\Rightarrow \quad S_1 \leq S_2$$

At same time S_2 is l.u.b and S_1 is other u.b.

$$\Rightarrow \quad S_2 \leq S_1$$

$$\Rightarrow \boxed{S_1 = S_2} \quad \blacksquare$$

Ex: $A, B \xrightarrow{f} \mathbb{R}$.

show that $f(A \cap B) \subseteq f(A) \cap f(B)$.

Proof:

$x \in A \cap B \iff x \in A$ and $x \in B$.

$f(A) = \{ f(x) : x \in A \}$

$$\Rightarrow \left.\begin{array}{l} f(x) \in f(A) \\ f(x) \in f(B) \end{array}\right\} f(x) \in f(A) \cap f(B)$$

$\Rightarrow$