Lecture 2: Completeness of $\mathbb{R}$

Rational numbers $\mathbb{Q} = \{ a/b \mid a, b \in \mathbb{N}, b \neq 0 \}$

we have $+, \cdot, <$ real numbers $\mathbb{R}$

we have $+$, $\cdot$, $<$

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$\mathbb{R}$ cannot be represented as $\mathbb{Q}$

$\sqrt{2} \notin \mathbb{Q}$, $\pi \notin \mathbb{Q}$.

today: what does it mean $\mathbb{R}$ is complete or has no holes.

Axiom of completeness:

Every non-empty subset of real numbers $A \subseteq \mathbb{R}$ that is bounded above has a least upper bound.

Doesn't hold for $\mathbb{Q}$.

Def: A set $A \subseteq \mathbb{R}$ is bounded from above if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$.

In this case, $b$ is called an upper bound of $A$. 
Example: \( I = (0,1) \subseteq \mathbb{R} \)

\( b = 1 \) - an upper bound b.c. \( a \in I \Rightarrow a < 1 \)

\( b = 7 \) - an upper bound b.c. \( a \in I \Rightarrow a \leq 7 \).

\( \Rightarrow \) upper bound is not unique.

\( I = (0, +\infty) \) - no upper bound

Def: A set \( A \subseteq \mathbb{R} \) is bounded from below if there exists a number \( \ell \in \mathbb{R} \) such that \( \ell \leq a \) for all \( a \in A \).

"\( \ell \) = lower bound of \( A \)"

Ex:

\( I = (0,1) \)

0 - is OK.

-1 - is OK

\( J = (-\infty, 0) \) - there are no lower bounds.
A real number \( \mathbb{R} \) is the least upper bound of a set \( A \subseteq \mathbb{R} \) if:

1. \( S \) is an upper bound.
2. If \( b \) is any other upper bound then \( S \leq b \).

Notation: \( S = \sup(A) \)

Example:

\[
I = (0, 1) \quad \sup(I) = 1
\]

\[
I = [0, 1] \quad \sup(I) = 1
\]

\[
I = \left\{ 1 - \frac{1}{n^2}, \; n=1,2,3,\ldots \right\} \quad \sup(I) = 1
\]

\[
I = \left\{ 0, \frac{1}{2}, \frac{1}{3}, \ldots \right\} \quad \sup(I) = 1
\]

\[
I = \left\{ 0, \frac{3}{4}, \frac{5}{7}, \ldots \right\} \quad \sup(I) = 1
\]

\( I \) is an upper bound.

Let \( b = \sup(I) \), assume \( b \neq 1 \)

Then \( b < 1 \) then we can find \( N \) large enough so that
\[ b < 1 - \frac{1}{N^2} \]

\[ \Rightarrow \text{Contradiction because } 1 - \frac{1}{N^2} \in I \]

but \[ 1 - \frac{1}{N^2} > b = \sup(I). \]

\[ \text{Def: } \text{"Greatest lower bound"} \]

- \[ \text{Def: } \text{Notation: } \inf(A) \]

\[ \text{Example: } I = \{0, 1\} \Rightarrow \inf(I) = 0. \]

\[ I = \{0, 1\} \Rightarrow \inf(I) = 0 \]

\[ I = [0, 1] \]

\[ \text{Def: } a_0 \in \mathbb{R} \text{ is a maximum } \]

of a set \( A \) if \( a_0 \) is an element of \( A \) and \( a_0 \geq a \) for all \( a \in A. \)

\[ \text{Def: } a \in \mathbb{R} \text{ is a minimum...} \]
Example:

\[ I = (0, 2) \implies \text{int}(I) = 0, \sup(I) = 2, \text{no max} \]
\[ I = [0, 2) \implies \text{int}(I) = 0, \sup(I) = 2, \text{no max} \]
\[ I = [0, 2] \implies \text{int}(I) = 0, \sup(I) = 2, \max(I) = 2. \]

\[ I = \{ x \in \mathbb{R} : x^2 < 2 \} = (-\sqrt{2}, +\sqrt{2}) \]

\[ \sup(I) = \sqrt{2} \]
\[ \inf(I) = -\sqrt{2} \]
\[ \min(I) = \text{DNE} \]
\[ \max(I) = \text{DNE} \]

**Axiom of completeness:**

Every non-empty subset of real numbers \( A \subseteq \mathbb{R} \), that is bounded above, has a least upper bound, \( S \in \mathbb{R} \).

\[ A = \{ x \in \mathbb{Q} : x^2 < 2 \} \subseteq \sqrt{2} \not\in \mathbb{Q} \]

\[ \Rightarrow \text{Doesn't hold for } \mathbb{Q} \]
\[ \text{but it holds in } \mathbb{R} \]
Ex: Prove that if $a$ is an element of $A$ and is also an upper bound of $A$ then it must be that $a = \text{sup}(A)$.

Proof: $A$ is bounded from above

$\Rightarrow$ By axiom of completeness there exists $b \in \mathbb{R}$ such that $b = \text{sup}(A)$.

$b$ is the least upper bound so, by def. of l.u.b. we know that $X \leq b \ \forall X \in A$.

$\Rightarrow$ in particular $a \leq b$.

As $a$ - upper bound

$b = \text{sup}(A)$

from definition of l.u.b. $b \leq a$ if $a$ is any other upper bound

$b \leq a$
Ex.: Prove that if a set $A \subseteq \mathbb{R}$ has a least upper bound that it is unique.

Proof: Assume a least upper bound of $A$ is not unique.

Let $S_1, S_2 \in \mathbb{R}$ so that $S_1 = \sup(A)$, $S_2 = \sup(A)$.

As $S_1$ is l.u.b and $S_2$ is other u.b

$\Rightarrow S_1 \leq S_2$

At same time $S_2$ is l.u.b and $S_1$ is other u.b.

$\Rightarrow S_2 \leq S_1$

$\Rightarrow \sqrt{S_1 = S_2}$
\[ \exists x : A, B \xrightarrow{f} \mathbb{R}. \]

Show that \( f(A \cap B) \subseteq f(A) \cap f(B) \).

**Proof:**

\[ x \in A \cap B \iff x \in A \text{ and } x \in B. \]

\[ f(A) = \{ f(x) : x \in A \} \]

\[ \Rightarrow f(x) \in f(A) \iff \begin{cases} f(x) \in f(A) \cap f(B) \end{cases} \]

\[ f(x) \in f(B) \]

\[ \Rightarrow \]

\[ \]