Lecture 1

Plan:
- Sets
- "If and only if"
- Induction on n
- Functions

① "Set = collection of elements"

Ex: $S = \{1, 2, 3\}$

$\mathbb{N} = \{1, 2, 3, \dots\}$ – natural numbers.

$\mathbb{Q} = \{\frac{a}{b},\ a, b \in \mathbb{N},\ b \neq 0\}$ – Rational numbers.

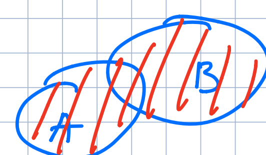
$\mathbb{R}$ – Real numbers

$S = \{\frac{1}{n},\ n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$S = \{x \in \mathbb{Q} : x^3 < 3\}$

$\emptyset$ – empty set – "Set with no elements".

② **Operations on sets:**



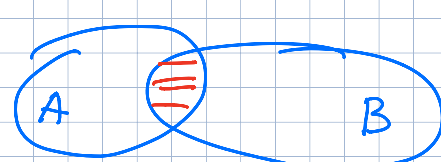
A, B – sets

"union of two sets"

$\Rightarrow$ $A \cup B = S \iff x \in S \iff x \in A$ or $x \in B$

"intersetion of sets $A \cap B$"

$A \cap B$; $x \in A \cap B \iff x \in A$ and $x \in B$



$x \in A_1 \cup A_2 \cup \dots \cup A_n \iff x \in A_1$ or $x \in A_2 \dots$ or $x \in A_n$

$x \in A_1 \cap A_2 \dots \cap A_n \iff x \in A_1$ and $x \in A_2 \dots$ and $x \in A_n$.

$$\boxed{\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup \dots}$$

notation for infinite union of sets.

$$\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \dots$$

Example: $A_n = \left\{ \frac{1}{m}, \quad m = n, n+1, n+2 \dots \right\}$

$A_1 = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$

$A_2 = \left\{ \frac{1}{2}, \frac{1}{3}, \dots \right\}$

$A_3 = \left\{ \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$

$$\bigcup_{n=1}^{\infty} A_n = A_1 \,;$$

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Proof: Assume that $B = \bigcap_{n=1}^{\infty} A_n$ is not empty.

If B is not empty then B contains at leas one element $\frac{1}{m} \in B$ for some $m \in \mathbb{N}$.

By definition of infinite intersection

$$\frac{1}{m} \in \bigcap_{n=1}^{\infty} A_n \iff \frac{1}{m} \in A_1, \ \frac{1}{m} \in A_2, \ \frac{1}{m} \in A_3 \ldots$$

then take $A_{m+1} = \left\{ \frac{1}{m+1}, \frac{1}{m+2}, \ldots \right\}$

obviously

$$\frac{1}{m} \notin A_{m+1}$$

$\Rightarrow$ Contradiction $\Rightarrow$ Assumption "$\bigcap_{n=1}^{\infty} A_n$ is not empty" is wrong.

$$\Rightarrow \bigcap_{n=1}^{\infty} A_n = \emptyset.$$

$\blacksquare$

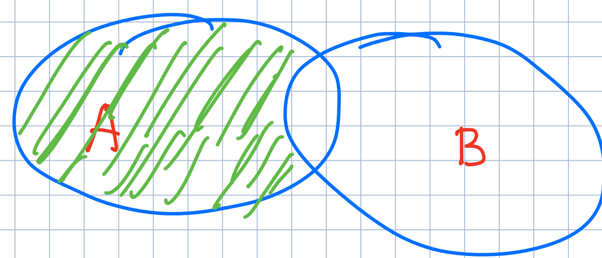
③ Inclusion Relation:

$$A \subseteq B \overset{\text{Def.}}{\iff} \text{if } x \in A \text{ then } x \in B$$

"A is a subset of the set B"

$$A \setminus B = \{ x \in A : x \in B \}$$

"complement of B in A"



(4) Proving theorems:

"if and only if" - condition.

Example: Prove that two real numbers $a, b \in \mathbb{R}$ are equal $a = b$ if and only if $|a - b| < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Proof: $(\Rightarrow)$ if $a = b$ then $|a - b| < \frac{1}{n}$, $n \in \mathbb{N}$.

$(\Leftarrow)$ if $|a - b| < \frac{1}{n}$, $\forall n \in \mathbb{N}$ then $a = b$.

($\Rightarrow$): if $a=b$ then $a-b=0$

$\Rightarrow |a-b|=0$ and $0<\frac{1}{n}$ $\forall n$.

$\Rightarrow |a-b|<\frac{1}{n}$ for all n.

($\Leftarrow$) Assume that $|a-b|<\frac{1}{n}$ $\forall n \in \mathbb{N}$ but $a \neq b$.

$n|a-b|<1$

Then, if $a \neq b \Rightarrow |a-b| = \varepsilon > 0$

but we can find a number N large enough

to have $\frac{1}{N} < \varepsilon$.

$\Rightarrow$ from assumption

$$|a-b| < \frac{1}{N} < \varepsilon$$

We conclude that $|a-b| = \varepsilon$

and $|a-b| < \varepsilon$ $\Big\}$ contradiction

$\Rightarrow a=b$ ∎

5 Proof by induction:

$Statement_1$, $Statement_2$, $Statement_3$, ...

Proof by induction:

1. Prove that from "$Statement_n$ = true" follows "$Statement_{n+1}$ = true".
2. Check that "$Statement_1$ = true"

Example: Prove that $n(n+1)$ is even number for $n = 1, 2, 3, \ldots$

$Statement_n$ = "$n(n+1)$ is even"

Induction step:

if $\underbrace{n(n+1) \text{ is even}}_{Statement_n}$ then $\underbrace{(n+1)(n+2) \text{ is even}}_{Statement_{n+1}}$

take a difference:

$$(n+1)(n+2) - n(n+1) = 2\cdot(n+1).$$

$$\Rightarrow (n+1)(n+2) = \underbrace{n(n+1)}_{\text{even}} + 2\cdot(n+1)$$

even

$\Rightarrow$ even.

(2) check ~~condi~~ Statement 1:

$n=1 \Rightarrow n(n+1) = 1\cdot 2$ - even.

Example:

let $y_1 = 6$ and $y_{n+1} = (2y_n - 6)/3$.

$$y_2 = \frac{(2\cdot y_1 - 6)}{3}$$

$$= \frac{2\cdot 6 - 6}{3} = \frac{6}{3} = \boxed{2}$$

Prove that $y_n > -6$ for $n = 1, 2, \ldots$

Solution:

Statement$_n$ = " $y_n > -6$ is true".

Induction step:

"Statement$_n$ true" $\Rightarrow$ "Statement$_{n+1}$ is true"

$y_n > -6 \Longrightarrow y_{n+1} > -6$

Assume $y_n > -6$. then $y_n = -6 + \epsilon$

some positive Real number.

$$y_{n+1} = \frac{(2y_n - 6)}{3} = \frac{2\cdot(-6+\epsilon) - 6}{3}$$

$$= \frac{-18 + 2\epsilon}{3} = -6 + \frac{2}{3}\epsilon$$ — positive

$\Rightarrow y_{n+1} > -6$ = " Statement$_{n+1}$ "

2) check Statement$_1$: $y_1 > -6$

But we kow that $y_1 = 6$.

<u>Def:</u> A function (a map) from a ~~A~~ set A to set B is a Rule f: which assignn to every element $a \in A$ an element $f(a) \in B$.