1 Equivariant elliptic cohomology

$X \circlearrowleft T \simeq (\mathbb{C}^\times)^n$

covariant functor:

$$\mathrm{Ell}_T : X \longrightarrow \mathrm{Ell}_T(X) \in \text{Schemes}.$$

Idea: $T = (\mathbb{C}^\times)^n$ act on $\mathbb{C}^n$
induces action on $X = \mathbb{P}(\mathbb{C}^n) = \mathbb{P}^{n-1}$

(a) Rational case

$$H_T(X) = \mathbb{C}[c, u_1, \dots u_n] \Big/ (c-u_1)(c-u_2)\dots(c-u_n)$$

$\mathrm{Spec}\big(H_T(X)\big)$ = union of n-hyperplanes
in ($\mathbb{C}^{n+1}$ with coordinates $c, u_1, \dots u_n$)
given by equations $c = u_i$

$$H_i = \{(c, u_1, \dots, u_n) : c = u_i\} \subset \mathbb{C}^{n+1}$$

$$\{H_i \text{ glued to } H_j \text{ at } u_i = u_j\} = \Delta$$

$$\boxed{\mathrm{Spec}\big(H_T(X)\big) = \Big(\coprod_{i=1}^{n} H_i\Big) \Big/ \Delta.}$$

(b) Trigonometric case

$$K_T(X) = \mathbb{C}[c^{\pm 1}, u_i^{\pm 1}, \dots u_n^{\pm 1}] \Big/ (c-u_1) \dots (c-u_n)$$

$$\Rightarrow \quad \mathrm{Spec}\,(H_T(X)) = \left(\coprod_{i=1}^{n} H_i \right) \Big/ \Delta$$

$$H_i = (\mathbb{C}^{\times})^n = \mathbb{C}^{\times}_{u_1} \times \dots \times \mathbb{C}^{\times}_{u_n}$$

$$\Delta = \{ H_i \text{ glued to } H_j \text{ at } u_i = u_j \}.$$

(c) Elliptic case: fix $E = \mathbb{C}^{\times} / q^{\mathbb{Z}}$

$$\mathrm{Ell}_T(X) = \left(\coprod_{i=1}^{n} H_i \right) \Big/ \Delta$$

$$H_i = E^n = E_{u_1} \times \dots \times E_{u_n}$$

$$\Delta = \{ H_i \text{ glued to } H_j \text{ at } u_i = u_j \}.$$

In general:

$$\boxed{\begin{aligned} \mathrm{Ell}_T(X) &= \left(\coprod_{p \in X^T}^{n} H_p \right) \Big/ \Delta \\ H_p &= E^{\dim(T)} \end{aligned}}$$

Extended version:

assume it is $\mathbb{Z}^{\dim(\mathrm{Pic})}$

$$\mathcal{E}_{Pic(X)} = Pic(X) \otimes E \simeq E^{\dim(\mathrm{Pic})}$$

then: $$\widehat{H}_p = \underbrace{E^{\dim(T)}}_{\text{equivariant parameters}} \times \underbrace{E^{\dim(\mathrm{Pic})}}_{\text{Kähler parameters}}$$

$$E_T(X) = \mathrm{Ell}_T(X) \times \mathcal{E}_{Pic(X)} = \left(\coprod_{p \in X^P} \widehat{H}_p \right) / \Delta$$

Def: Elliptic cohomology class is $(s, \mathcal{L})$

$\mathcal{L}$ - line bundle over $E_T(X)$

s - section of $\mathcal{L}$.

Explicitly:

$$s = (s_1, \dots, s_N) \qquad s_i = s\big|_{\widehat{H}_i}$$

such that $$s_i\big|_{\widehat{H}_i \cap \widehat{H}_j} = s_j\big|_{\widehat{H}_i \cap \widehat{H}_j}$$

Elliptic Stable envelope (M. Aganagic, A. Okounkov)

$p \in X^T$; $\sigma \in \mathrm{cochar}(T)$

$\mathrm{Stab}_\sigma(p)$ - Elliptic cohomology class uniquely defined by a set of axioms.

$$T_{p,q} = \mathrm{Stab}_\sigma(p)\Big|_{\mathcal{A}_q}$$

(A_1) quasiperiods of $T_{p,q}$ in all variables

(A_2) $T_{p,q}$ is triangular in Bruhat order determined by σ.

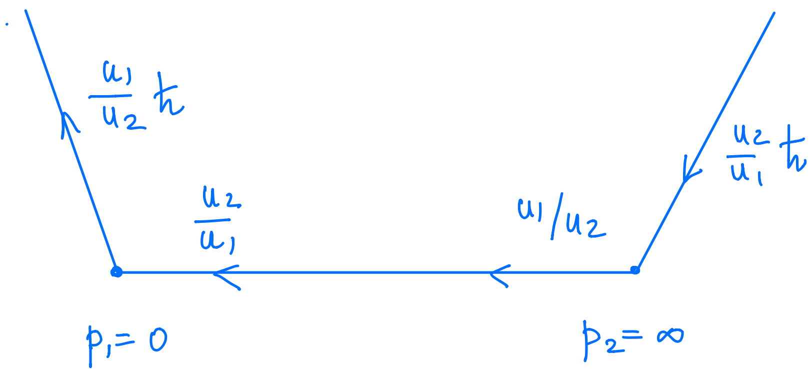
(A_3)
$$T_{p,p} = \prod_{\substack{w \in \mathrm{char}_T(T_p X) \\ \langle w, \sigma \rangle < 0}} \vartheta(w).$$

Example:

$$X = T^*_{\hbar}\mathbb{P}^1$$

$$T = \mathbb{C}^\times_{\hbar} \times \mathbb{C}^\times_{u_1} \times \mathbb{C}^\times_{u_2}$$

$T^*\mathbb{P}^1 =$



$\mathrm{Pic}(X) \otimes E \simeq E$ with coordinate z.

$$T^{\sigma}_{p,q} = \begin{pmatrix} \vartheta\left(\hbar \frac{u_1}{u_2}\right), & \vartheta(\hbar)\, \dfrac{\vartheta(z\, u_1/u_2)}{\vartheta(z)} \\ 0 & \vartheta(u_1/u_2) \end{pmatrix}$$

"Half of $E_{\hbar,q}(\widehat{\mathfrak{sl}}_2)$ R-matrix"

coincide at $u_1 = u_2$

as $\vartheta(1) = 0$

Example: $X = T^*\mathbb{P}^1$, $X^T = \{p_1 = 0, p_2 = \infty\}$.

(From the representation theory of $U_q(\widehat{sl}_2)$

$T^*\mathbb{P}^1 \sim (\mathbb{C}^2 \otimes \mathbb{C}^2)[0]$; $p_1 = \uparrow\otimes\downarrow$, $p_2 = \downarrow\otimes\uparrow$)

$$\mathrm{Stab}_c(p_1) = \vartheta(x u_2 z)\,\vartheta(x u_1 \hbar)$$

$$\mathrm{Stab}_c(p_2) = \vartheta(x u_2)\,\vartheta(x u_1 z \hbar)$$

(These are elliptic "weight functions" of A. Varchenko for $[\mathbb{C}^2 \otimes \mathbb{C}^2][0]$

elliptic analogue of Bethe vectors)

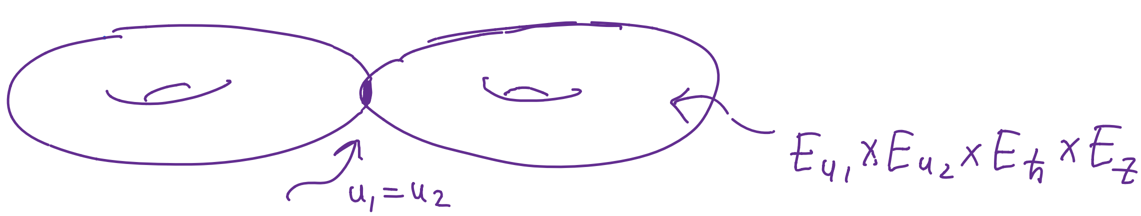
$$\mathrm{Stab}_c(p_i)\Big|_{\hat{O}_{p_j}} = \text{substitution of } x = \frac{1}{u_j}$$

(classical solution of Bethe equation)

$u = u_1/u_2$

$$\widetilde{T} = \begin{pmatrix} \vartheta(\hbar u)\,\vartheta(z), & \vartheta(\hbar)\,\vartheta(u z) \\ 0 & \vartheta(u)\,\vartheta(\hbar z) \end{pmatrix}$$

$u_1 = u_2$



Symplectic duality (3D-mirror symmetry).

"Resonances"
Felder-Varchenko

$u = u_1/u_2$

$$\widetilde{T} = \begin{pmatrix} \vartheta(\hbar u)\,\vartheta(z) & \vartheta(\hbar)\,\vartheta(uz) \\ 0 & \vartheta(u)\,\vartheta(\hbar z) \end{pmatrix}$$

$z = 1$

$u = 1$

3D Mirror symmetry for $X = T^*\mathbb{P}^1$

Invariance under $u \longleftrightarrow z$

$$T^!(u,z) = T(z,u)$$

Geometric picture

$X = T^*\mathbb{P}^1$ $\quad\quad\quad$ $X^! = T^*\mathbb{P}^1$

$\circlearrowright$ $\quad\quad\quad$ $\circlearrowright$

$T = \mathbb{C}_u^\times \times \mathbb{C}_\hbar^\times$ $\quad\quad\quad$ $T^! \simeq \mathbb{C}_z^\times \times \mathbb{C}_\hbar^\times$

$K = Pic(X) \otimes \mathbb{C}^\times = \mathbb{C}_z^\times$ $\quad\quad\quad$ $K^! = \mathbb{C}_u^\times$

$$Stab^X(p)\big|_q = Stab^{X^!}(q)\big|_p.$$

General Case

$X \circlearrowleft A \times \mathbb{C}_\hbar^\times$ $\quad\quad\quad$ $X^! \circlearrowleft A^! \times \mathbb{C}_\hbar^\times$

$K = Pic(X) \otimes \mathbb{C}^\times$ $\quad\quad\quad$ $K^! = Pic(X^!) \otimes \mathbb{C}^\times$

Definition of 3D-mirror symmetry:

1. Bijection $X^A \simeq X^{!\,A^!}$
2. isom: $\varkappa: A \simeq K^!$, $K \simeq A^!$
3. $$T^X_{p,q} = \varkappa^*\left(T^{X^!}_{q^!,p^!}\right)$$

We expect that 1, 2, 3 are equivalent to symplectic duality in other senses.

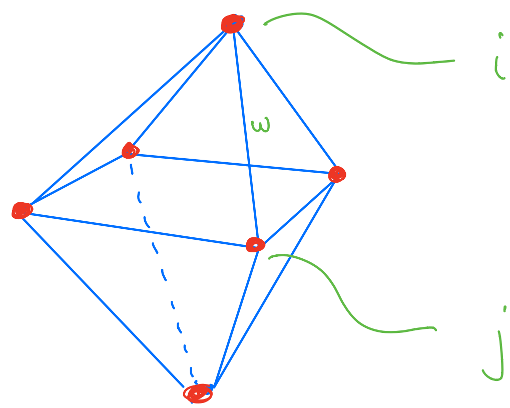
Example:

$$X = T^*_\hbar Gr(2,4) \qquad \mathbb{C}^2(u_1)\otimes\mathbb{C}^2(u_2)\otimes\mathbb{C}^2(u_3)\otimes\mathbb{C}^2(u_4)$$

"2 spins up"

$$T = \mathbb{C}^\times_\hbar \times \mathbb{C}^\times_{u_1}\times\mathbb{C}^\times_{u_2}\times\mathbb{C}^\times_{u_3}\times\mathbb{C}^\times_{u_4}, \quad K = \mathbb{C}^\times_z$$

$Gr(2,4) =$

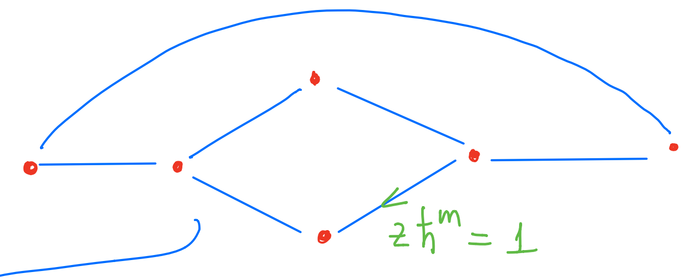


Elliptic envelope

$$T^\times = \begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

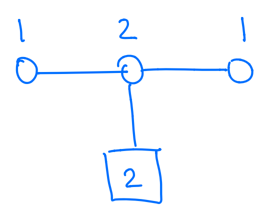
Equal at $z\hbar^m$ for some $m \in \mathbb{Z}$?

$w = 1$

$X^! =$ 

equivariant skeleton of quiver variety

associated with quiver



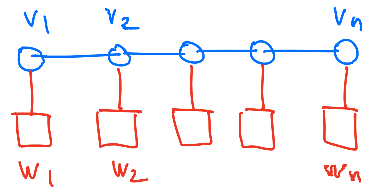
We proved several types of such identities in

arXiv : 1902.03677 for $X = T^* Gr(k,n)$

arXiv : 1906.00134 for $X = T^*\left(\text{Complete Flags in } \mathbb{C}^N\right)$

General prediction for $X^!$

X = quiver variety:



$$\lambda = \sum v_i \alpha \quad ; \quad \mu = \sum w_i \, \omega_i^*$$

Then, the symplectic dual variety is a "slice" in affine grassmannian: OR "monopole moduli space"

$$Gr = G((z)) / G[[z]]$$

$$Gr = \bigsqcup_{\lambda \in \text{dominant weights}} Gr^\lambda \qquad Gr^\lambda = G[[z]] \cdot t^\lambda$$

$$G_1 = \{ g(z) \in G([[z]]) : \lim_{z \to \infty} g(z) = 1 \}$$

$$\boxed{X^! = Gr^\lambda_\mu = \overline{Gr^\lambda} \cap G_1 \cdot t^\mu}$$

Duality interface (mother function)

$$Y = \mathrm{Ell}_{T\times T^!}(X \times X^!) \ni m$$

then $\quad p \in X^T, \quad q \in (X^!)^{T^!}$

we have equivariant embeddings:

$$\begin{array}{c} \mathrm{Ell}_{T\times T'}(p \times X^!) \xhookrightarrow{i_p} Y \\ \wr\wr \\ \mathrm{Ell}_T(p) \times \mathrm{Ell}_{T'}(X') = E_{T'}(X') \end{array}$$

So in total:

$$E_T(X) \xhookrightarrow{i_q} Y \xhookleftarrow{i_p} E_{T'}(X')$$

The conjecture is that

$$\mathrm{Stab}_X(q) = i_q^*(m)$$

$$\mathrm{Stab}_{X'}(p) = i_p^*(m)$$

Then, the coincidence of matrix elements is trivial:

$$
\begin{tikzcd}
 & m \arrow[dl, "|_p"'] \arrow[dr, "|_q"] & \\
\mathrm{Stab}^{X}(p) \arrow[d, "|_q"'] & & \mathrm{Stab}^{X'}(q) \arrow[d, "|_p"] \\
T^{X}_{pq} \arrow[rr, phantom, "="] & & T^{X'}_{qp}
\end{tikzcd}
$$