QDE, monodromy and 3D-mirror symmetry

I Vertex functions:

$G \circlearrowright \mathbb{C}^n \Rightarrow X = T^*\mathbb{C}^n ////_\theta G$

$\theta \in \mathbb{R}^{rk(G)}$ - stability parameter for GIT

$T \circlearrowright X$ - torus acting on X, X^T - finite.

$$\boxed{V_p(z) = \sum_d \# \left\{ \begin{array}{l} \text{rational curves } \mathbb{P}^1 \subset X \\ \text{passing through } p \in X^T \\ \text{of degree } d \end{array} \right\} z^d}$$

More precisely: moduli spaces of quasimaps

$$QM_p^d = \{ \mathbb{P}^1 \overset{f}{\dashrightarrow} X : f(\infty) = p \in X^T, \deg(f) = d \}$$

$\circlearrowleft \mathbb{C}_q^*$

$$V_p(z) = \sum_{d \in H_2(X,\mathbb{Z})} \chi(QM_p^d)\, z^d$$

$z^d = z_1^{d_1} \cdots z_R^{d_R}$ – "Kähler parameters" (Dynamical)

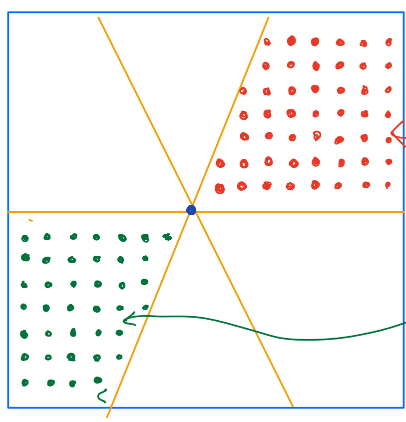
$\chi(QM_p^d) \in \mathbb{Q}(\underbrace{a_1, \ldots a_n, \hbar, q}) $ "equivariant parameters"

In examples $V_p(z)$ are some interesting q-hypergeometric series with finite radius of convergence.

How it depends on stability parameter?

$$V_p(z) = \sum_{d \in \mathrm{Cone}_\theta \subset H_2(X,\mathbb{Z})} C_d(a)\, z^d$$

defined by choice of stability θ.



$\mathbb{R}^{2k(G)}$

Stability condition θ_+
vertex functions are holomorphic near $z_i \sim 0$

Stability conditions θ_-
vertex functions are holomorphic near $z_i = \infty$

Important idea:

(1) $V_p(z)$, $p \in X^T$ - form a basis of solutions of certain q-difference equation

(2) This $q$de doesn't depend on the choice of stability θ

Monodromy: $\boxed{V_p^{\theta_1}(z) = \sum \mathrm{Mon}_{p,q}(z)\, V_q^{\theta_2}(z)}$

Example (oversimplified)

H. Dinkins
arXiv: 1912.04834

 $\rightsquigarrow \quad X_{\pm\theta} = T^*\mathbb{P}^0 \simeq$ point

For vertex functions with stability conditions we can find:

$$V_{+\theta} = \sum_{n=0}^{\infty} \frac{(\hbar)_n}{(q)_n} z^n = \prod_{n=0}^{\infty} \frac{1-\hbar z q^n}{1-zq^n}$$

holom near $z=0$

$$(x)_n = (1-x)\dots(1-q^{n-1}x)$$

holom. near $z=\infty$.

$$V_{-\theta} = \sum_{n=0}^{\infty} \frac{(\hbar)_n}{(q)_n} z^{-n} \left(\frac{q}{\hbar}\right)^n = \prod_{n=1}^{\infty} \left(\frac{1-q^n/z}{1-q^n/\hbar z}\right)$$

Both function solve the q-difference equation:

$$F(zq) = \frac{1-z}{1-\hbar z} F(z)$$

operator $M(z)$ from last time.

The monodromy is given by elliptic function:

$$\mathrm{Mon}(z) = \frac{V_{+\theta}}{V_{-\theta}} \simeq \frac{\vartheta(\hbar z)}{\vartheta(z)}$$

$$\vartheta(z) = \prod_{n=0}^{\infty} (1-zq^n)\left(1-\frac{q^{n+1}}{z}\right)$$

Jacobi theta function.

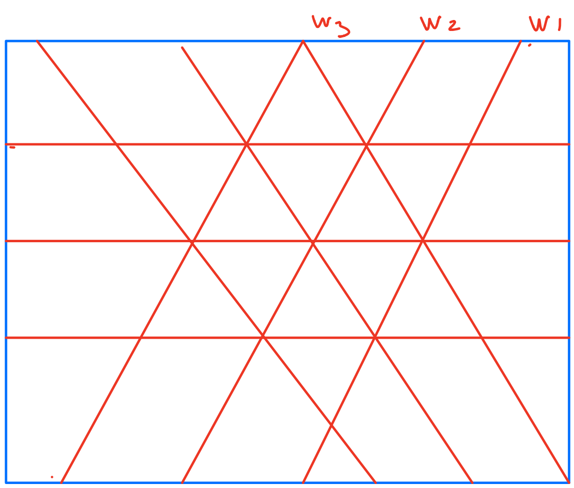
The main problem: compute qde for arbitrary X

What do we know about qde already?
arXiv: 1602.09007 (A. Okounkov, A.S.)

qde's are of the form:

1
$$\Psi(zq^{\mathcal{L}}) = M_{\mathcal{L}}(z)\,\Psi(z), \quad \mathcal{L} \in \mathrm{Pic}(X)$$
$$M_{\mathcal{L}}(z) \in \mathrm{End}\,(K_T(X))\,(z).$$

2
$$M_{\mathcal{L}}(z) = B_{w_1}(z) \cdot \ldots \cdot B_{w_n}(z)$$



← $\mathrm{Pic}(X)$-periodic arrangement of hyperplanes called "walls"

$B_w(z)$ - "wall crossing" operators, which has poles only at:
$$z = q^w$$

$$\boxed{\text{Main problem: understand } B_w(z)}$$

In particular, the monodromy has similar factorization:

$$B_w \sim \frac{1}{1 - z q^{-w}}$$

$$\mathrm{Mon}(z) = \overrightarrow{\prod_{w \in \mathrm{Walls}}} B_w(z)$$

Central idea: one can construct $B_w(z)$ from the limit of monodromy:

$$\boxed{\lim_{q \to 0} \mathrm{Mon}(z q^w) \sim \ldots B_w(z)}$$

Essentialy, only one factor of contribute in limit

If we know the monodromy then we can reconstruct $B_w(z) \Longrightarrow$ qDE $\Longrightarrow$ vertex functions

Back to our prime example:

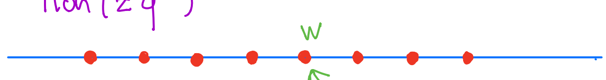
$$\mathrm{Mon}(z) = \frac{V_{+\theta}}{V_{-\theta}} \simeq \frac{\vartheta(\hbar z)}{\vartheta(z)}$$

$$\vartheta(z) = \prod_{n=0}^{\infty} (1 - z q^n)\left(1 - \frac{q^{n+1}}{z}\right)$$

For $w \in \mathbb{Q}$ we have:

$$\underbrace{\lim_{q \to 0} \frac{\vartheta(z \hbar q^w)}{\vartheta(z q^w)}}_{\mathrm{Mon}(z q^w)} = \begin{cases} \hbar^{-\lfloor w \rfloor - 1/2}, & w \notin \mathbb{Z} \\ \dfrac{1 - z\hbar}{1 - z} \hbar^{-w - 1/2}, & w \in \mathbb{Z}. \end{cases}$$

$B_w(z)$.



limit to the wall

Monodromy from 3D-mirror symmetry:

$$V_p(z) = \sum_{d \in H^2(X,\mathbb{R})} c_d(a)\, z^d$$

holom in z near $z=0$

typically have poles $a \sim q^i$, $i=1,2,\dots$ accumulating near $a=0$

What about a basis of solutions holom in a near $a=0$?

Theorem [Aganagic-Okounkov, 1604.00423]

1. There exist a basis of solutions $V_p^!$ holomorphic near $a \sim 0$.

2. Transition matrix

$$V_p^! = \sum_{q \in X^T} U_{p,q}(a,z)\, V_p(z)$$

is the matrix of the "elliptic stable envelope" in the basis of fixed points

$$U_{p,q} = \mathrm{Stab}_\sigma^{Ell}(p)\Big|_q, \quad p,q \in X^T$$

$$\mathrm{Stab}_\sigma^{Ell}(p) \in \mathrm{Ell}_T(X)$$

called "elliptic stable envelope of p".

What is the geometric meaning of $V_p^!$?

Conjecture [3D-mirror symmetry]

1. There exists a "3D-mirror" variety $X^!$ (symplectic dual of X) together with

$z \longleftrightarrow a$

In words: equivariant and kähler parameters are exchanged by mirror symmetry

2. $$V_p^! = \sum c_d(z)\, a^d$$

– vertex functions of $X^!$

Note that: a is Kähler parameter of $X^!$
z is equivariant for $X^!$

Elliptic stable envelope of X

= transition matrix from basis of solutions $V_p^X(z)$

to basis of solutions $V_p^{X^!}(a)$

Monodromy from mirror symmetry:

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Vertex functions}\\ \text{for } X_{\theta_+}\\ \text{holom. near } z\sim 0\end{array}} \arrow[rr, "\mathrm{Stab}^{X^!}_{\theta_-}\left(\mathrm{Stab}^{X^!}_{\theta_+}\right)^{-1}"] & & \boxed{\begin{array}{c}\text{Vertex functions}\\ \text{for } X_{\theta_-}\\ \text{holom. near } z\sim \infty\end{array}} \\
& \boxed{\begin{array}{c}\text{Vertex functions}\\ \text{of mirror } X^!\\ \text{holom near } a\sim 0\end{array}} \arrow[ul, "\mathrm{Stab}^{X^!}_{\theta_+}"] \arrow[ur, "\mathrm{Stab}^{X^!}_{\theta_-}"'] &
\end{tikzcd}
$$

In this way we obtain

$$\boxed{\mathrm{Mon}(z) = \left(\mathrm{Stab}^{X^!}_{+\theta}\right)^{-1} \circ \left(\mathrm{Stab}^{X^!}_{-\theta}\right) = R_{X^!}(z,a)}$$

this object is called "elliptic dynamical R-matrix of $X^!$"

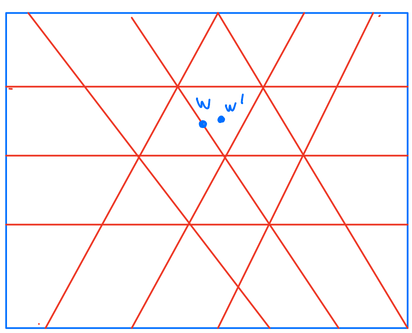
This gives explicit formulas for the monodromy of qde of X (enumerative geometry of X) in terms of elliptic cohomology of $X^!$ (algebraic topology of $X^!$)

Back to wall crossing operators:

$$U_{p,q}(z) = \mathrm{Stab}^{Ell}_{\theta}(p)\big|_q$$

Theorem [Y. Kononov, A.S.]:

if $w \in H^2(X,\mathbb{R})$ is on the wall
and $w' \in H^2(X,\mathbb{R})$ small deformation of w



then the limit factorize:

$$\lim_{q\to 0} U(zq^{w}, a) = A^{w'}(a)\, Z(z)$$

depends only on a

depends only on z.

① $A^{w'}_{p,q}(a) = \mathrm{Stab}^{Kth,w'}_{X}(p)\big|_q$

② $Z_{p,q}(a) = \mathrm{Stab}^{Kth,w}_{Y_w}(p)\big|_q$

$w \longrightarrow$ finite group $\mu_w = \langle e^{2\pi i w}\rangle \circlearrowright X^!$

$Y_w = (X^!)^{\mu_w} \subset X^!$

μ_w - fixed subvariety in the mirror.

The main result:

$$B_w(z) = \lim_{q\to 0} \mathrm{Mon}(z q^w) = A_{w'}^{-1} Z_+^{-1} Z_- A_{w'}$$

R_{Y_w}

$= K$ theoretic R-matrix of the subvariety $Y_w \subset X^!$

In representation theory R_{Y_w} is "trigonometric R-matrix" of some quantum group which acts on $K_T(Y_w)$

$$\boxed{\begin{array}{l}\textbf{Theorem:}\\ \text{In the stable basis of } K_T(X) \text{ with slope } w', \text{ the matrix of the wall}\\ \text{crossing operator } B_w(z) \text{ coincides with trigonometric } R\text{-matrix}\\ \text{of the quantum group associated with } Y_w \text{ in representation } K_T(Y_w).\end{array}}$$

"Dynamical Weyl group" of X = "R-matrices of $Y_w \subset X^!$"

Example 1 $\quad U_q(\widehat{gl}_n) - U_q(\widehat{gl}_m)$ duality.

$X = A_n$ - type quiver variety

Y_w, $X^!$ - A_m - type quiver varieties.

$B_w(z) \sim$ R-matrices of $U_q(\widehat{gl}_m)$

$$M(z) = B_{w_1}(z) \ldots B_{w_n}(z) \sim R_{i,n-1} R_{i,n-2} \ldots R_{i,1}$$

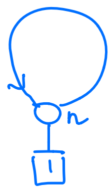
And the quantum difference equation for X

$$\Psi(z q^{\mathcal{L}}) = M(z)\Psi(z)$$

Turns to quantum Knizhnik-Zamolodchikov equation for $U_q(\widehat{gl}_m)$

$\boxed{\text{Dynamical equations for } U_q(\widehat{gl}_n)}$ $\xleftrightarrow{\text{3D-mirror symmetry}}$ $\boxed{\text{qKZ equations for } U_q(\widehat{gl}_m)}$

Example 2 Toroidal algebras



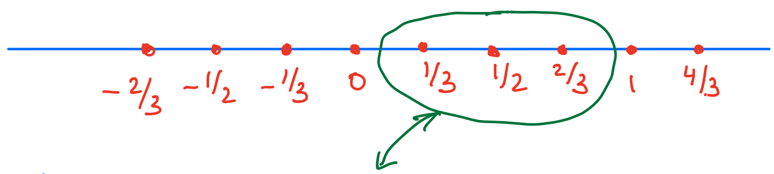
$\Rightarrow \quad X = \mathrm{Hilb}^n(\mathbb{C}^2)$

$K_T(X) \circlearrowleft U_{q,t}(\widehat{\widehat{gl}}_1)$

(Elliptic Hall algebra, Ding-Iohara-Miki algeba ....)

$H^2(X,\mathbb{R}) = \mathbb{R}$, $\quad \text{Walls} = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; |b| \leq n \right\}$

$n=3$

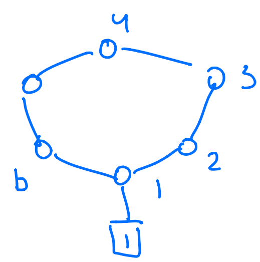


$M(z) = B_{1/3}(z)\, B_{1/2}(z)\, B_{2/3}(z)$

3D Mirror : $X^! \cong X$

$w = \frac{a}{b} \Rightarrow \mu_w = \langle e^{2\pi i\, a/b} \rangle \hookrightarrow X^!$

$Y_w = (X^!)^{\mu_w} =$



Cyclic quiver of length b

$K_T(Y_w) \circlearrowleft U_{q,t}(\widehat{\widehat{gl}}_b)$ - quantum toroidal algebras

$M(z) \sim R_{U_{q,t}(\widehat{\widehat{gl}}_3)}\, R_{U_{q,t}(\widehat{\widehat{gl}}_2)}\, R_{U_{q,t}(\widehat{\widehat{gl}}_3)}$ [Generalized qKZ]