GDE, monodromy and 3D-mirror symmetry

I  Vertex functions:

\[ G \subset \mathbb{C}^n \Rightarrow X = T^*\mathbb{C}^n // B G \]

\[ \Theta \in \mathbb{R}^{2k(G)} \text{ - stability parameter for GIT} \]

\[ T \cong \mathbb{T} \text{ - torus acting on } X, X^T \text{ - finite} \]

\[ V_p(z) = \sum_d \# \left\{ \text{rational curves } \mathbb{P}^1 \rightarrow X \text{ passing through } p \in X^T \text{ of degree } d \right\} \cdot z^d \]

More precisely: moduli spaces & quasimaps

\[ \mathbb{Q} M_p^d = \left\{ \mathbb{P}^1 \rightarrow X : f(\infty) = p \in X^T, \deg f = d \right\} \]

\[ V_p(z) = \sum_{d \in H_2(X, \mathbb{Z})} X(\mathbb{Q} M_p^d) \cdot z^d \]

\[ z^d = z_1^d \cdots z_r^d \text{ - } "Kähler parameters" \text{ (Dynamical)} \]

\[ X(\mathbb{Q} M_p^d) \in \mathbb{Q}(a_1, \ldots, a_k, q) \text{ - } "equivariant parameters" \]
In examples $V_p(z)$ are some interesting $q$-hypergeometric series with finite radius of convergence.

How it depends on stability parameter?

$$V_p(z) = \sum_{d \in \text{Cone}_\Theta \subset \mathbb{C}_2(x,z)} C_d(a) z^d$$

defined by choice of stability $\Theta$.

Important idea:

1. $V_p(z), p \in X^\tau$ — form a basis of solutions of certain $q$-difference equation

2. This q-d.e. doesn't depend on the choice of stability $\Theta$

Monodromy:

$$V_p^\Theta(z) = \sum \text{Mon}_{p,q}(z) V_q^\Theta(z)$$
Example (oversimplified)

\[ \sim \quad X_{\pm \theta} = T^* \mathbb{P}^* \approx \text{point} \]

For vertex functions with stability conditions we can find:

\[ V_{+\theta} = \sum_{n=0}^{\infty} \left( \frac{q}{q_n} \right)^n z^n = \prod_{n=0}^{\infty} \frac{1 - \frac{1}{2} z q^n}{1 - z q^n} \]

\( (x)_n = (1-x) \cdots (1-q^{n-1} x) \)

\[ V_{-\theta} = \sum_{n=0}^{\infty} -\frac{1}{(q)_n} z^n (\frac{q}{x})^n = \prod_{n=1}^{\infty} \frac{1 - q^{n/2}}{1 - q^{n/2} z} \]

Both functions solve the q-difference equation:

\[ F(z q) = \frac{1 - z}{1 - \frac{1}{2} q z} \cdot F(z) \]

The monodromy is given by elliptic function:

\[ \text{Mon}(z) = \frac{V_{+\theta}}{V_{-\theta}} \sim \frac{\mathcal{G}(z)}{\mathcal{G}(z)} \]

\[ \mathcal{G}(z) = \prod_{n=0}^{\infty} (1 - z q^n) (1 - \frac{q^{n+1}}{z}) \quad \text{Jacobi theta function.} \]
The main problem: compute $qde$ for arbitrary $X$

What do we know about $qde$ already?


$qde's$ are of the form:

1. $\Psi(z, q^L) = M_{\xi}(z) \Psi(z), \quad L \in \text{Pic}(X)$
   $M_{\xi}(z) \subseteq \text{End}(K_{\tau}(X))(z)$.

2. $M_{\xi}(z) = B_{w_1}(z) \cdots B_{w_n}(z)$

Pic($X$)-periodic arrangement of hyperplanes called "walls"

$B_{w}(z)$ - "wall crossing" operators, which has poles only at:

$z = q^{w}$

Main problem: understand $B_{w}(z)$
In particular, the monodromy has similar factorization:

$$\text{Mon}(z) = \prod_{w \in \text{Walls}} B_w(z)$$

Central idea: one can construct $B_w(z)$ from the limit of monodromy:

$$\lim_{q \to 0} \text{Mon}(z q^w) \sim \ldots B_w(z)$$

Essentially, only one factor of contribute in limit

If we know the monodromy, then we can reconstruct $B_w(z) \Rightarrow \text{qDE} \Rightarrow \text{vertex functions}$

Back to our prime example:

$$\text{Mon}(z) = \frac{V_{\text{ref}}}{V_0} \sim \frac{\mathcal{C}(z^z)}{\mathcal{C}(z)}$$

$$\mathcal{C}(z) = \prod_{k=0}^{\infty} (1-z q^w) (1-z q^{w'})$$

For $w \in \mathbb{Q}$, we have:

$$\lim_{q \to 0} \frac{\mathcal{C}(z q^w)}{\mathcal{C}(z q^{w'})} = \left\{ \begin{array}{ll}
\frac{1}{q^{w^{1/2}}} & , \quad w \notin \mathbb{Z} \\
\frac{1-2z}{1-z} \frac{1}{q^{w^{1/2}}} & , \quad w \in \mathbb{Z}.
\end{array} \right.$$
Monodromy from 3D-mirror symmetry:

$$V_p(z) = \sum_{d \in \Omega^d(X, \mathbb{R})} C_d(a) \cdot z^d$$

holom in $z$ near $z=0$

typically have poles $a \sim q^i$

$i=1, 2, \ldots$

accumulating near $a=0$

What about a basis of solutions holom in $a$ near $a=0$?

Theorem [Aganagic-Okoanov, 1604.00423]

1. There exist a basis of solutions $V_p^i$ holomorphic near $a=0$.

2. Transition matrix

$$V_p^i = \sum_{q \in \mathcal{X}^+} U_{p, q} (a, z) \cdot V_p (z)$$

is the matrix of the "elliptic stable envelope" in the basis of fixed points

$$U_{p, q} = \text{Stab}_{\sigma}^E(p) \big|_q$$

$p, q \in \mathcal{X}^+$

$$\text{Stab}_{\sigma}^E(p) \in E_{\mathbb{C}}(X)$$

called "elliptic stable envelope at p"
What is the geometric meaning of $V_p$?

Conjecture [3D-mirror symmetry]

1. There exists a "3D-mirror" variety $X'$ (symplectic dual of $X$) together with

$$z \leftrightarrow a$$

In words: equivariant and Kähler parameters are exchanged by mirror symmetry

2. $V_p^i = \sum c_i(z) a^d$

- vertex functions of $X'$

Note that: $a$ is Kähler parameter of $X'$.
$z$ is equivariant for $X'$.

Elliptic stable envelope of $X$

= transition matrix from basis of solutions $V_p^X(z)$ to basis of solutions $V_p^{X'}(a)$
**Monodromy from mirror symmetry.**

**Vertex functions** for $X_{\Theta^+}$ holom. near $z \approx 0$

**Vertex functions** for $X_{\Theta^-}$ holom. near $z \approx 0$

\[ \text{Stab}_{\Theta^+}^{X^!} \left( \text{Stab}_{\Theta^-}^{X^!} \right)^{-1} \]

**Vertex functions** of mirror $X^!$ holom near $a \approx 0$

In this way we obtain

\[ \text{Mon}(z) = \left( \text{Stab}_{\Theta^+}^{X^!} \right)^{-1} \circ \left( \text{Stab}_{\Theta^-}^{X^!} \right) = R_{X^!}(z, a) \]

This object is called "elliptic dynamical $R$-matrix at $X^!".

This gives explicit formulas for the monodromy at qde of $X$ (enumerative geometry of $X$) in terms of elliptic cohomology of $X^!$ (algebraic topology of $X^!$)
Back to wall crossing operators:

$$U_{p,q}(z) = \text{Stab}^{Ell}_{\vartheta}(p) \big|_q$$

**Theorem [Y. Kononov, A.S.]:**

If $w \in H^2(X,\mathbb{R})$ is on the wall and $w' \in H^2(X,\mathbb{R})$ small deformation of $w$ then the limit

$$\lim_{q \to 0} U(zq^w, a) = A^{w'}(a) \cdot Z(z)$$

depends only on $a$.

1. $A^{w'}_{p,q}(a) = \text{Stab}^{Kth, w'}_X(p) \big|_q$

2. $Z_{p,q}(a) = \text{Stab}^{Kth, w}_{Y_w}(p) \big|_q$

$w \to$ finite group $\mu_w = \langle e^{2\pi i w} \rangle \subset X!$

$Y_w = (X!)^{\mu_w} \subset X!$ $\mu_w$ - fixed subvariety in the mirror.
The main result:

\[ B_w(z) = \lim_{q \to 0} \text{Mon}(z q^w) = A_w^{-1} \begin{bmatrix} Z_+ & Z_- \end{bmatrix} A_w \]

\[ R_{Y_w} = K \text{ theoreic } R\text{-matrix} \]

at the subvariety \( Y_w \subset X \).

In representation theory \( R_{Y_w} \) is "trigonometric R-matrix" at some quantum group which acts on \( K_T(Y_w) \).

Theorem:

In the stable basis at \( K_T(X) \) with slope \( w' \), the matrix at the wall crossing operator \( B_w(z) \) coincides with trigonometric R-matrix at the quantum group associated with \( Y_w \) in representation \( K_T(Y_w) \).

"Dynamical Weyl group" = "R-matrices at" \( X \)

\( Y_w \subset X \)!
Example 1 \( U_q(\widehat{g_{\ln}}) - U_q(\widehat{g_{\ln}}) \) duality.

\[ X = A_{n-1} \text{-type quiver variety} \]

\[ Y_{w_j} X^j = A_{m-1} \text{-type quiver varieties.} \]

\[ B_{w_j}(z) \sim R \text{-matrices at } U_q(\widehat{g_{\ln}}) \]

\[ M(z) = B_{w_1}(z) ... B_{w_n}(z) \sim R_{i,n-1} R_{i,n-2} ... R_{i,1} \]

And the quantum difference equation for \( X \)

\[ \Psi(z, q^z) = M(z) \Psi(z) \]

Turns to quantum Knizhnik-Zamolodchikov equation for \( U_q(\widehat{g_{\ln}}) \)

Dynamical equations for \( U_q(\widehat{g_{\ln}}) \)

3D-mirror symmetry

qKZ equations for \( U_q(\widehat{g_{\ln}}) \)
Example 2  Toroidal algebras

\[ X = \text{Hilb}^n(\mathbb{C}^2) \]

\[ K_T(X) \cong U_{q,t}(\widehat{\mathfrak{g}_n}) \]

(Elliptic Hall algebra,  
Ding-Iohara-Miki algebra ...)

\[ H^2(X, \mathbb{R}) = \mathbb{R}, \quad \text{Walls} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid |b| \leq n \right\} \]

\[ n=3 \quad \bullet \quad -\frac{1}{3} \quad -\frac{1}{2} \quad -\frac{1}{3} \quad \circ \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad i \quad \frac{1}{3} \]

\[ M(z) = B_{\frac{1}{3}}(z) B_{\frac{1}{2}}(z) B_{\frac{2}{3}}(z) \]

3D Mirror: \( X' \cong X \)

\[ w = \frac{a}{b} \Rightarrow \mu_w = \langle e^{2\pi i \frac{a}{b}} \rangle \subseteq X' \]

\[ Y_w = (X')^{\mu_w} = \]

Cyclic quiver of length \( b \)

\[ K_T(Y_w) \cong U_{q,t}(\widehat{\mathfrak{g}_n}) - \text{quantum toroidal algebras} \]

\[ N(z) \sim \mathbb{C}[U_{q,t}(\widehat{\mathfrak{g}_3}) U_{q,t}(\widehat{\mathfrak{g}_2}) U_{q,t}(\widehat{\mathfrak{g}_3}) \left[ \begin{array}{c} \text{Generalized} \end{array} \right] \quad q \in \mathbb{Z} \]