In the previous talk:

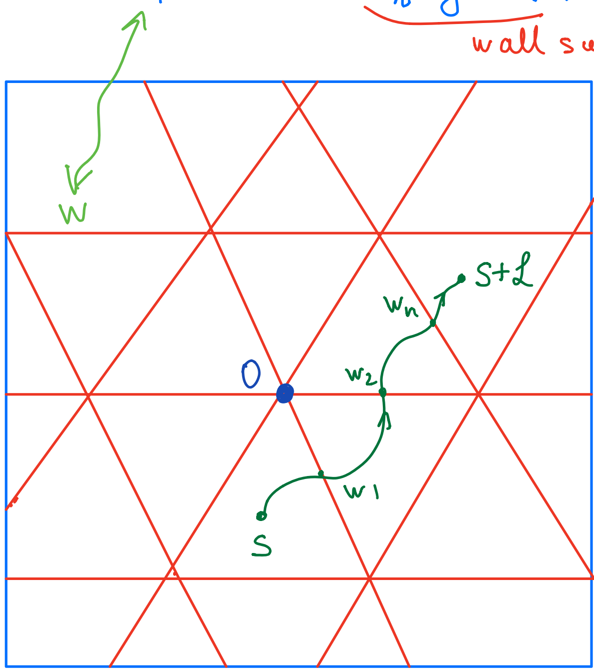
vertex functions of quiver variety X

$$V(a, z q^{\mathcal{L}}) = M_{\mathcal{L}}(z, a)\, V(a, z)$$

The problem is to describe:

$$M_{\mathcal{L}}(z) = B_{w_1}(z)\, B_{w_2}(z) \ldots B_{w_n}(z)$$

$$B_{w_1}(z) \in \underbrace{U_{\hbar}(\mathfrak{g}_w)}_{\text{wall subalgebra.}}(z) \circlearrowright \oplus K(X)$$

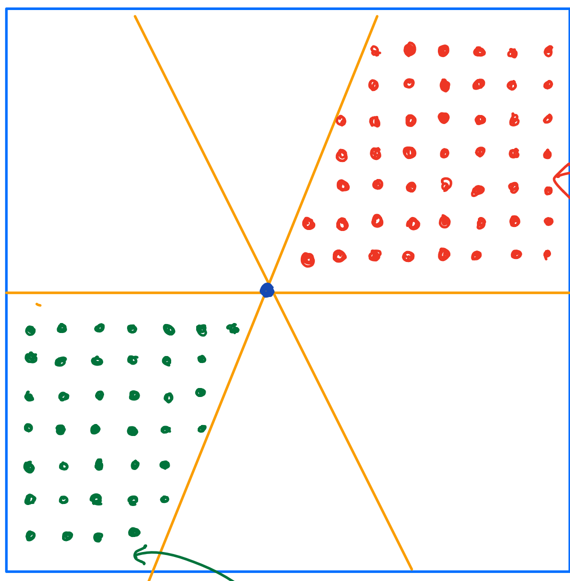


Pic (X) - periodic arrangement of hyperplanes in $H^2(X, \mathbb{R})$.

Vertex functions of quiver varieties:

$X_\theta = T^*R /\!\!/_\theta G$; $\theta \in \mathbb{R}^{2k(G)}$ - stability parameter.

$$\text{Vertex function} = \sum_{d \in \mathrm{Cone}_\theta} c_d(a)\, z^d \in K_T(X).$$



Stability condition θ_+
Vertex function is holom. $z_i \sim 0$

Stability condition θ_-.
Vertex function is holom near $z_i \sim \infty$.

The idea: The q-difference equations stay invariant under the change of θ.

$$\Longleftrightarrow$$

Vertex functions for $X_{+\theta}$ and $X_{-\theta}$ solve the same qde's.

Monodromy operator = transformation from one basis of solutions to another:

$$V_{+\theta}(a,z) = \mathrm{Mon}(a,z)\, V_{-\theta}(a,z)$$

q-periodic transition matrix of monodromy

At the same time, the fundamental solutions of qde's for $X_{+\theta}$ and $X_{-\theta}$

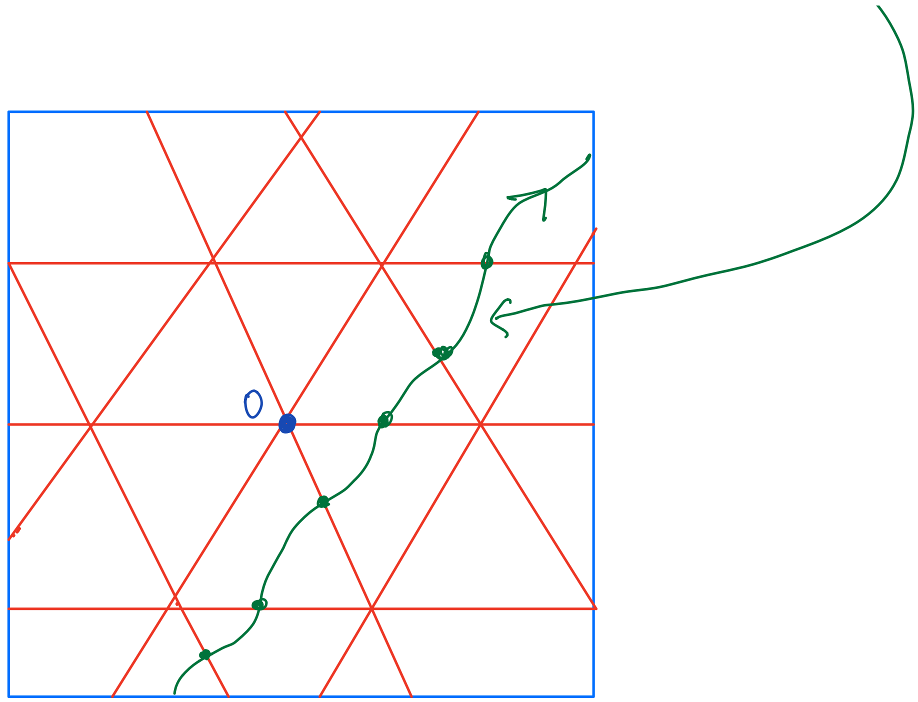
$$\Psi_\pm(zq) = M^\pm(z)\, \Psi_\pm(z)$$

could be choosen in the form:

$$\Psi_\pm(z) \sim M^\pm(z)^{-1} M^\pm(zq)^{-1} M^\pm(zq^2) \dots$$

and the monodromy is:

$$\mathrm{Mon} = \Psi_+(z) \circ \Psi_-^{-1}(z) \sim \overleftarrow{\prod_{\substack{\text{all walls} \\ \text{in } H^2(X,\mathbb{R}) \\ \text{along some} \\ \text{path}}}} B_w(z)$$



Idea: Invent a limit $q \to 0$ in which only one term of the infinite product survive:

$$\lim_{w} \mathrm{Mon}(z) \longrightarrow B_w(z)$$

This would provide new description of wall-crossing operators $B_w(z)$ if we can fix the monodromy $\mathrm{Mon}(z)$ from geometry of X.

Very simple example:

$$\begin{tikzcd} \textcircled{1} \\ \boxed{1} \arrow[u] \end{tikzcd} \rightsquigarrow X_{\pm\theta} = T^*\mathbb{P}^0 \simeq \text{point}$$

For vertex functions with stability conditions we can find:

$$V_{+\theta} = \sum_{n=0}^{\infty} \frac{(\hbar)_n}{(q)_n} z^n = \prod_{n=0}^{\infty} \frac{1-\hbar z q^n}{1-zq^n}$$

holom near $z=0$

$$(x)_n = (1-x)\cdots(1-q^{n-1}x)$$

holom. near $z=\infty$.

$$V_{-\theta} = \sum_{n=0}^{\infty} \frac{(\hbar)_n}{(q)_n} z^{-n}\left(\frac{q}{\hbar}\right)^n = \prod_{n=1}^{\infty}\left(\frac{1-q^n/z}{1-q^n/\hbar z}\right)$$

Both function solve the q-difference equation:

$$F(zq) = \frac{1-z}{1-\hbar z} F(z)$$

operator $M(z)$ from last time.

The monodromy is given by elliptic function:

$$\mathrm{Mon}(z) = \frac{V_{+\theta}}{V_{-\theta}} \simeq \frac{\vartheta(\hbar z)}{\vartheta(z)}$$

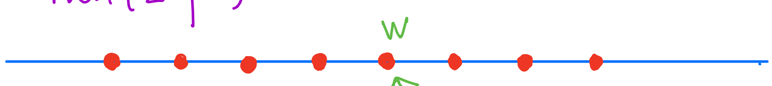
$$\vartheta(z) = \prod_{n=0}^{\infty} (1 - z q^n)\left(1 - \frac{q^{n+1}}{z}\right)$$ *Jacobi theta function.*

We can find $M(z)$ from the limit of monodromy: For $w \in \mathbb{Q}$:

$$\lim_{q \to 0} \frac{\vartheta(z \hbar q^w)}{\vartheta(z q^w)} = \begin{cases} \hbar^{-\lfloor w \rfloor - 1/2}, & w \notin \mathbb{Z} \\ \dfrac{1 - z\hbar}{1 - z}\, \hbar^{-w - 1/2}, & w \in \mathbb{Z}. \end{cases}$$

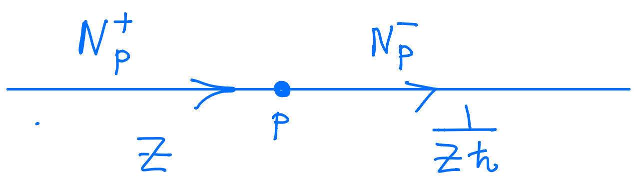
$$\mathrm{Mon}(z q^w)$$

$M_w(z)$.



limit to the wall

The symplectic dual $X^! \simeq \mathbb{C}^2$ with torus action: $K = \mathbb{C}^\times \otimes \mathrm{Pic}(X)$:



A.-O. elliptic stable envelopes for $X^!$

and the monodromy is symply

$$\text{Monodromy} \simeq \frac{\vartheta(\hbar z)}{\vartheta(z)} = \frac{\Theta(N_p^-)}{\Theta(N_p^+)} = \frac{\mathrm{Stab}_-^{X^!}(p)}{\mathrm{Stab}_+^{X^!}(p)}.$$

Symplectic duality:

$X, T, K \qquad\qquad X^!, T^!, K^!$

① $T \simeq K^!$, $K \simeq T^!$ } Kähler $\longleftrightarrow$ equivariant parameters

② $X^T \simeq (X^!)^{T^!}$ – dual varieties have same fixed points.

③ q-difference equations for X and $X^!$ are the same.

$$V^X(a,z) = \sum_{d \geq 0} c_d(a) z^d \; ; \quad V^{X^!}(a,z) = \sum_{d \geq 0} c_d^!(z)\, a$$

holom. in z near $z=0$

holom. in a near $a=0$.

different solutions to same QDE.

Thus: $V^{X^!}(a,z) = \boxed{S(a,z)}\, V^X(a,z)$

Some q-periodic operator

Theorem (M. Aganagic, A. Okounkov)

$$V^{X^!}(a,z) = \mathrm{Stab}^X(a,z)\, V^X(a,z)$$

"Elliptic stable envelope of X"

In basis of fixed points:

$$\begin{pmatrix} * \\ * \\ * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & 0 & * & * & * \\ & & & * & * \\ & & & & * \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ * \\ * \end{pmatrix}$$

q-hypergeom. functions holom in z

Sections of certain line bundle over $Ell_{T\times K}(pt) = E^n$

q-hypergeom functions holom. in a

Note, that this implies also that

$$\boxed{\mathrm{Stab}^X(a,z) = \mathrm{Stab}^{X^!}(a,z)^{-1}}$$

Checked in arXiv: 1902.03677 ; 1906.00134

Monodromy from stable envelope

$$
\begin{tikzcd}
 & V^{X^!}(a,z) \arrow[dl, "\mathrm{Stab}^{X^!}_{+\theta}"'] \arrow[dr, "\mathrm{Stab}^{X^!}_{-\theta}"] & \\
V^{X}_{+\theta}(a,z) \arrow[rr, "\mathrm{Stab}^{X^!}_{+\theta} \circ \left(\mathrm{Stab}^{X^!}_{-\theta}\right)^{-1}"'] & & V^{X}_{-\theta}(a,z)
\end{tikzcd}
$$

Monodromy = elliptic R-matrix of the symplectic dual variety $X^!$

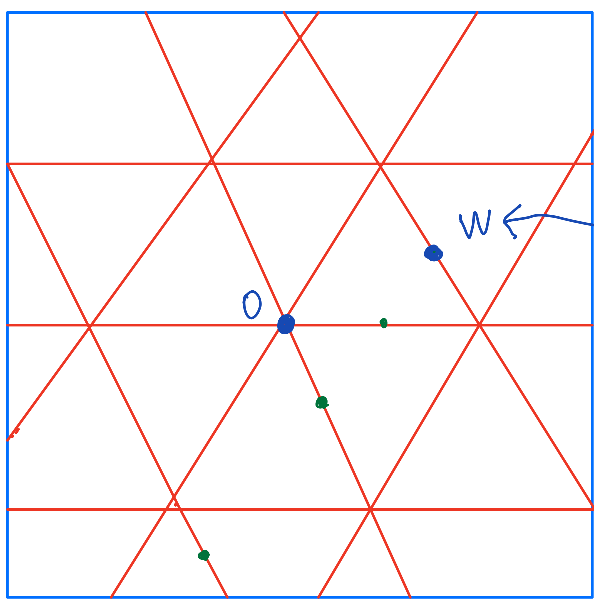
$$
\boxed{\mathrm{Mon}(z) = \mathrm{Stab}^{X^!}_{+\theta} \circ \left(\mathrm{Stab}^{X^!}_{-\theta}\right)^{-1}}
$$

Main Result: (Yakov Kononov, S.)

arXiv: 2004:07862

Let $w = (w_1, \ldots, w_m)$ with $w_i \in \mathbb{Q}$

be a point on a wall:



Denote

$$z q^{w} = (z_1 q^{w_1}, \ldots, z_m q^{w_m})$$

Then we have a "limit to the wall"

(1):

$$\lim_{q \to 0} \mathrm{Stab}^{\mathcal{E}ll}(z q^{w}, a) = Z\, A$$

$$\begin{pmatrix} * & * & * \\ & * & * \\ 0 & & * \end{pmatrix}$$

upper triangular matrices.

$A_{ij} \in \mathbb{Q}(q, \hbar)$

$Z_{ij} \in \mathbb{Q}(z, \hbar)$

2. A_{ij} = K-theoretic stable envelope of X with slope w in the basis of fixed point.

3. Z_{ij} = K-theoretic stable envelope of $Y_w^! \subset X^!$ with slope 0 in the basis of fixed points.

4. Let $\mu_w \subset T^!$ denotes a cyclic subgroup generated by:

$$(z_1 \dots z_m) \longrightarrow (z_1 e^{2\pi i w_1}, \dots z_m e^{2\pi i w_m})$$

then $Y_w^! = (X^!)^{\mu_w}$.

Corollary:

$$B_w(z) \sim \lim_{q\to 0} \mathrm{Mon}(zq^w)$$

$$= \lim_{q\to 0} \mathrm{Stab}_+^{X^!}(zq^w)^{-1} \circ \mathrm{Stab}_-^{X}(zq^w)$$

$$= A^{-1} \underbrace{Z_+^{-1} Z_-}_{\text{K-theoretic R-matrix of } Y_w^!} A$$

$\Longleftrightarrow$

Informaly, this means that:

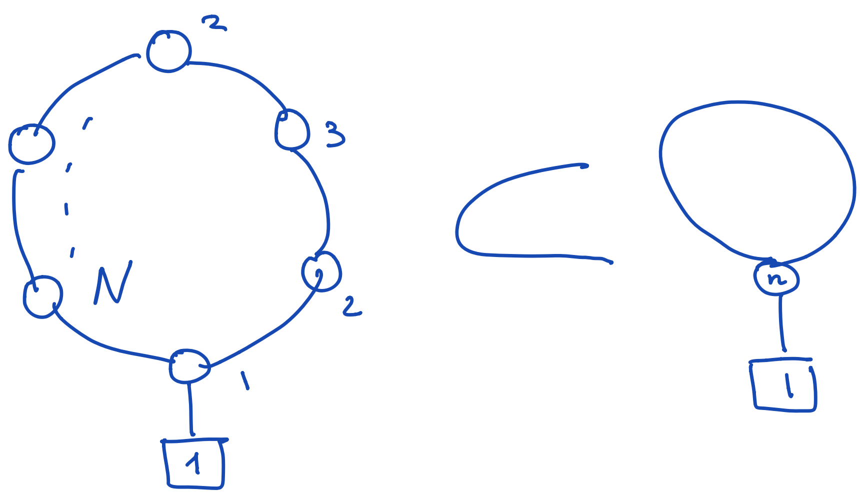
The matrix of the wall-crossing operator $B_w(z)$ of variety X in the stable basis with slope w coincides with K-theoretic R-matrix of $Y_w^!$ with slope 0 in the basis of fixed points.

Example: $X \simeq \mathrm{Hilb}^n(\mathbb{C}^2) \simeq X^!$

In this case if $w = \frac{p}{N} \in \mathbb{Q}$ then

$\mu_w : a \to a e^{2\pi i \frac{p}{N}}$ and:

$X^{\mu_w} = Y^!_w$ = cyclic quiver variety



$\Rightarrow$ the set of walls is exactly

$$\left\{ \frac{p}{N} \in \mathbb{Q} \;\middle|\; 1 \le |N| \le n \right\}$$

The K theory of the quiver variety $Y^!_w$ is equipped with an action of $U_{t_1,t_2}(\widehat{\mathfrak{gl}}_N)$.

Thm: (Y. Kononov, S.)

The wall crossing operator $B_{\frac{s}{N}}(z)$ for $\mathrm{Hilb}^n(\mathbb{C}^2)$ in the stable basis of slope s/N coincides with the R-matrix of the quantum toroidal algebra $U_{t_1,t_2}(\widehat{\widehat{gl}}_N)$ in the basis of fixed points.