Motivation:

quiver variety

$$QM^d = \left\{ \mathbb{P}^1 \overset{p_0 \qquad p_\infty}{=} \circlearrowleft \mathbb{C}_q^\times \dashrightarrow^{d} X = T^*V /\!\!/_{A} G \circlearrowleft T = A \times \mathbb{C}_\hbar^\times \right\}$$

$d \in H^2(X, \mathbb{Z})$.

Constructed Ciocan-Fontanine, Kim, Maulik (2014).

$$\exists \ \mathcal{O}_d^{vir} \in K(QM^d)$$

$$ev_p : \begin{array}{ccc} QM^d & \longrightarrow & X \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ f & \longrightarrow & f(p) \end{array}$$

$$V(a,z) = \sum_d ev_{p*}(\mathcal{O}_d^{vir})\, z^d \in K_{T \times \mathbb{C}_q}(X)_{loc}[[z]]$$

Rational function in equivariant param $(a_1, \ldots, a_n, \hbar)$

Kähler parameters.

From general theory you know that

$$V(aq^{\sigma}, z) = A_\sigma(a,z)\, V(a,z)$$

$$V(a, zq^{\mathcal{L}}) = M_{\mathcal{L}}(a,z)\, V(a,z).$$

$\sigma \in \mathrm{cochar}(A)$

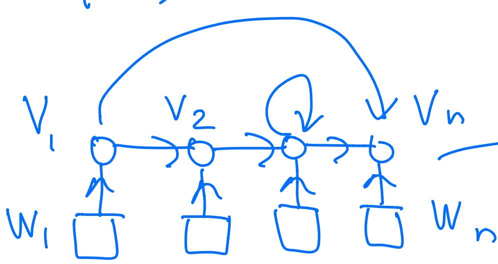
$\mathcal{L} \in \mathrm{cochar}(K) = \mathrm{Pic}(X).$

$K = \mathrm{Pic}(X) \otimes \mathbb{C}^\times$

__Question:__ Find a good description of $A_\sigma(a,z),\ M_{\mathcal{L}}(a,z) \in \mathrm{End}(K_T(X))$.

__Part I__ Representation theory

(A) Quiver varieties:



$$X = T^*_\hbar \mathrm{Rep}_Q /\!/_{\theta} \prod_i GL(V_i)$$

$$\mathrm{Rep}_Q = \bigoplus_{i \to j} \mathrm{Hom}(V_i, V_j)$$

① X is equipped with action

$$T = A \times \mathbb{C}^\times_\hbar$$

$A = $

// max torus of $\left(\prod_i GL(W_i)\right)$.

(2) X is equipped with tautological bundles.

$\mathcal{V}_i$ of ranks $\dim(V_i)$

$\mathcal{V}_i$ generate $K_T(X)$.

$\mathcal{L}_i = \det \mathcal{V}_i$ generate $\mathrm{Pic}(X) = \mathbb{Z}^{|Q|}$.

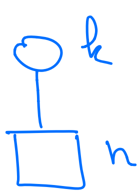
(3) tensor product structure:

$$\bigoplus_{V=(\dim V_i)} K_T(X^A)_{loc} \cong F_{i_1} \otimes F_{i_2} \cdots \otimes F_{i_n}$$

F_i = "fundamental representations"

$$= \bigoplus_V K_T\left(X\left(\text{with } W_k = \delta_{ik}\right)\right)$$

Example:



$\longrightarrow$ $X = T^*_\hbar Gr(k,n)$

$A = (\mathbb{C}^\times)^n$

$\mathbb{C}^\times_\hbar$

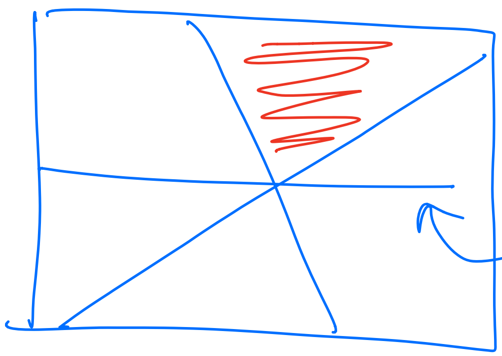
$$\bigoplus_{k=0}^{n} K_T(T^*Gr(k,n)) \cong \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n\text{-times}}.$$

(B) K-theoretic stable envelope

(Maulik-Okounkov)

$$K_T(X^A) \xrightarrow{\mathrm{Stab}_C^S} K_T(X)$$

C = "chamber" $\subset \mathrm{Lie}_{\mathbb{R}}(A)$

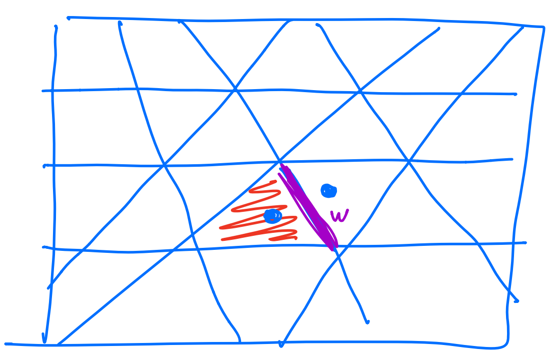


$\mathrm{Lie}_{\mathbb{R}}(A)$

$w^{\perp}$

w - weights $N_{X^A/X}$

S = "slope" $\in \mathrm{Lie}_{\mathbb{R}}(K) = \mathrm{Pic}(X) \otimes \mathbb{R}$.

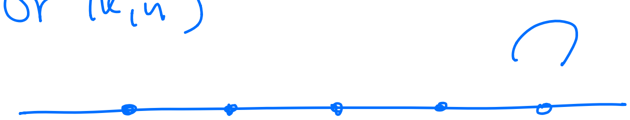


$\mathrm{Lie}_{\mathbb{R}}(K)$

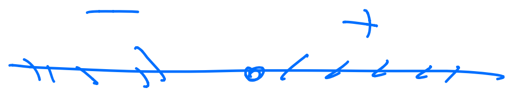
$\mathrm{Pic}(X) = \mathbb{Z}^{|Q|}$

periodic arrangement

for $T^*Gr(k,n)$ $\qquad \mathbb{Z}\langle \mathcal{O}(1) \rangle$.

$\mathrm{Lie}_{\mathbb{R}}(K)$  $\mathbb{R}$

(c) R- matrices



Assume $A = \mathbb{C}^{\times} \implies C = \{-,+\}$

Def: Total R-matrix

$$R^{(s)} = \left(\mathrm{Stab}_{-}^{s}\right)^{-1}\left(\mathrm{Stab}_{+}^{s}\right) \in \mathrm{End}(\underbrace{K_T(X^A)}_{F_1\otimes\cdots\otimes F_n})$$

Def: Wall R-matrix:

$s_1\ s_2$ slopes separated by a single wall

$$R_w^{\pm} = \left(\mathrm{Stab}_{\pm}^{s_2}\right)^{-1}\left(\mathrm{Stab}_{\pm}^{s_1}\right) \in \mathrm{End}(F_1\otimes\cdots).$$

Thm:

For all s and w the R-matrices $R^{(s)}$ and $R_w^{\pm}$ satisfy QYBE: in $F_1\otimes F_2\otimes F_3$:

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}.$$

"constant" R-matrix

"R-matrix with spectral parameter" = equiv param A).

(D) qKZ-equation = q-difference equations in equivariant par. a.

let $V(a,z)$ valued in $\underbrace{F_{i_1} \otimes \cdots \otimes F_{i}} \simeq \bigoplus_{v} K(x)$

Thm:

$$V(a_1, \ldots a_i q \ldots a_n \vec{z}) = A_i(\vec{a}, z) V(a z)$$

$$A_i(a,z) = \text{"} R_{i,i-1} \ldots R_{i,1} \, z^{v} \, R_{i,n}^{-1} \ldots R_{i,i+1}^{-1} \text{"}$$

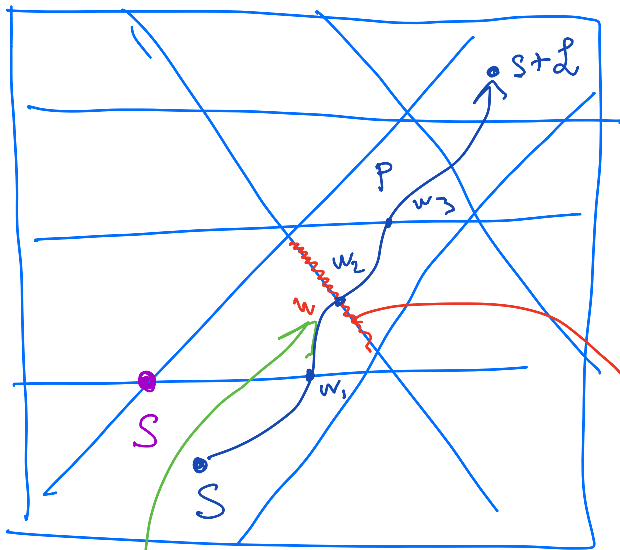
$$z^{v} = \mathrm{diag}\,(z^{v_i}, \ldots)$$

this system of equations is compatible.

(I. Frenkel - K. Reshetikhin ~1992).

(E) Faddeev-Reshetikhin-Takhtadzhyan:

Collection of vector spaces $\{F_1 \ldots F_n\}$ + collection of $R_{F_iF_j}$ satisfying QYBE $\rightsquigarrow$ Hopf algebra $(U(g), \Delta, S, m)$ acting in F_i.



$\mathbb{R} \otimes \mathrm{Pic}(X) = \mathbb{R}^{|Q|}$

$U_\hbar(\hat{\mathfrak{g}}_Q) \circlearrowright \bigoplus_v K(X)$

$\cup$

$U_\hbar(\mathfrak{g}_w)$

∃ unique triangular solution of ABRR equation.

$J(z) \in U_\hbar(\mathfrak{g}_w)^{\otimes 2}(z)$

$J_w(zq)\, z^\vee R^+_w = z^\vee J_w(z)$

Wall-crossing operator:

$$B_w(z) = m\left(S \otimes 1 \left(J_w(z)\right)\right)$$

multiplication of $U_\hbar(\mathfrak{g}_w)$

antipode

$\cap$

$U_\hbar(\mathfrak{g}_w)(z)$

$\circlearrowright$

$\bigoplus_v K_T(X)$

$$\boxed{M_{\mathcal{L}}(z) = \mathcal{L}\, B_{w_1}(z) \cdots B_{w_n}(z)}$$

<u>Thm:</u> The system of q-difference equations

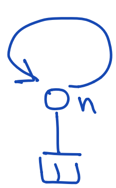
$$V(a, zq^{\mathcal{L}}) = M_{\mathcal{L}}(z)\, V(a,z)$$

(1) is compatible

(0) $M_{\mathcal{L}}(z)$ doesn't depend on the choice of path.

(2) is compatible with qKZ-equation.

<u>Example:</u>



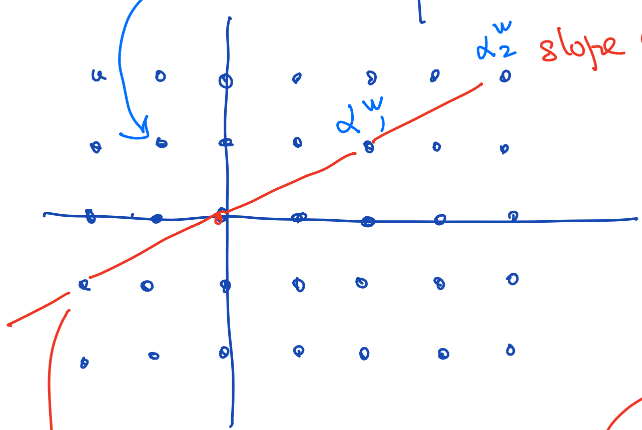
$\rightsquigarrow X_n = \mathrm{Hilb}^n(\mathbb{C}^2)$

$$\bigoplus_{n=0}^{\infty} K_T(X_n) \simeq \mathrm{Fock} = \mathbb{C}[p_1, p_2, \ldots].$$

$$U_\hbar(\mathfrak{g}_Q) \simeq U_\hbar(\widehat{\widehat{\mathfrak{gl}(1)}}) \approx \text{Elliptic Hall algebra}$$

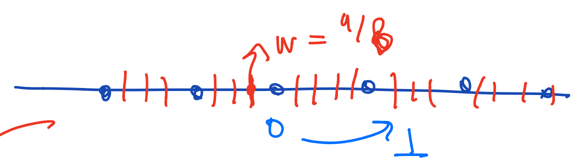
$$\simeq SH_{\mathfrak{gl}(\infty)}.$$

$$\alpha_{mn} = \{\alpha_{n,m} \mid m,n \in \mathbb{Z}\} / \text{Relations}$$



Wall subalgebras

$\mathrm{Pic}(X) \otimes \mathbb{R} = \mathbb{R}$



$\mathrm{Pic}(X) = \mathbb{Z}$

$$\text{Walls} = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; 1 \leq b \leq n \right\}$$

Heisenberg subalgebra in $\widehat{gl}(1)$

$$B_w(z) = :\exp\left(\sum_{k=0}^{\infty} \frac{\alpha^w_{-k}\, \alpha^w_k}{1 - z^{bk} q^{ak}} \right):$$

$$\boxed{M_{\mathcal{O}(1)} = \mathcal{O}(1) \prod_{\substack{w \in \mathbb{Q} \\ 0 < w < 1}} B_w(z)}$$

finite product.

Comment: cohomological limit

$M_{\mathcal{O}(1)} \rightsquigarrow \nabla_{\mathrm{Hilb}(\mathbb{C}^n)}$

in $DH_T(X_n)$ found by

Okounkov – Pandharipande

arXiv 0906.3587.